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On the Daksh-Majumder-Balaji Geometry

Daksh Jain, Soham Majumder, Amitesh Balaji, Daksh Saksena, Amitesh

Abstract- We introduce the Daksh-Majumder-Balaji (DMB) geometry, a pro-posed noncommutative geometric model for the quantum vacuum. The construction is defined by a Golden-ratio spectral dimension DDMB = 3 + ϕ−2, a fractional Dirac operator, and a ϕ−2-scaled deformation of the Moyal product on spacetime. We formulate the associated spectral triple, define the metric structure, and derive the foundational quantum mechanical framework using the Dirac–von Neumann axioms expressed on the Hilbert space HD . A preliminary formulation of the spectral ac-tion is presented, establishing the pathway toward extracting gravitational and gauge dynamics. This work provides the structural basis for future physical predictions arising from the DMB spectral geometry.

Keywords- Euclidean Geometry, Non-Euclidean Geometry, Coordinate Geometry, Analytic Geometry, Projective Geometry, Differential Geometry

I. INTRODUCTION

In this work, we introduce a new noncommutative spectral geometry, which we call the Daksh-Majumder-Balaji (DMB) geometry. The key novelty is the identification of a Golden-ratio spectral dimension

$$D_{DMB} = 3 + \phi^{-2},$$

together with a fractional Dirac operator and a ϕ−2 scaled Moyal deformation of spacetime. These elements generalize the Connes-Chamseddine spectral triple while preserving its core structure. Because the spectral dimension modifies the ultraviolet scaling of operators, the behaviour of quantum fields at high energy
is altered. Thus, the DMB geometry is proposed as a candidate model for the quantum vacuum with potentially improved ultraviolet behaviour relevant to quantum gravity.

II. DEFINITIONS

Define the Universal Spectral Dimension, or the Daksh–Majumder–Balaji in-variant, DDMB, as

$$D_{DMB} = 3 + \phi^{-2} = 5 - \phi.$$

The physical relevance of DDMB appears through heat-kernel asymptotics of the fractional Dirac operator. Specifically,

$$\mathrm{Tr}\left(e^{-tD^2_{DMB}}\right) \sim t^{-D_{DMB}/2} \qquad (t \to 0),$$

so that the ultraviolet scaling of propagators and the coefficients in the spectral action are governed directly by DDMB.

Here,

$$\phi = \frac{1+\sqrt{5}}{2}.$$

Define the Daksh–Majumder–Balaji geometry, or the DMB geometry, as a met-ric space. Assume this space has a Hausdorff dimension dimH(DMB) = DDMB and a global metric gD.

Where,

$$\dim_H(X) = \inf\{d \geq 0 : \mathcal{H}^d(X) = 0\}$$

Define the Daksh–Majumder–Balaji geometry, or the DMB geometry, as a met-ric space. Assume this space has a Hausdorff dimension dimH(DMB) = DDMB and a global metric gD.

Where,

$$\dim_H(X) = \inf\{d \geq 0 : \mathcal{H}^d(X) = 0\}$$

And the Hausdorff d-dimensional outer measure is:

$$\mathcal{H}^d(X) = \lim_{\delta \to 0} \mathcal{H}^d_\delta(X)$$

$$\mathcal{H}^d_\delta(X) = \inf\left\{\sum_{i=1}^{\infty}(\mathrm{diam}(U_i))^d : \bigcup_{i=1}^{\infty} U_i \supseteq S, \mathrm{diam}(U_i) < \delta\right\}$$

Where:

$$S \subset X, d \in [0, \infty)$$

Assume that the DMB geometry represents a quantum vacuum, and along with it, define the non-commutative geometry definition of the DMB geometry by a Spectral Triple:

- Algebra AJ is a generalized, non-commutative algebra below the Planck scale, such that [x, p]≠ iℏ. It is a non-commutative ring of observables.
- Hilbert space $\mathrm{H}_{D,}$D, where all the quantum states of the quantum vacuum DMB exist.
- Non-commutative Dirac operator DDMB.

The Metric

Assume AJ is represented on HD as a C∗-algebra. Then define the metric gD as a global metric on the DMB space. Note that gD is a metric, but not a tensor in the usual differential-geometric sense.

$$g_D(\phi, \psi) = d(\phi, \psi) = \sup\{|\phi(a) - \psi(a)| : a \in \mathcal{A}_J, ||[\mathcal{D}_{DMB}, a]|| \leq 1\}$$

Where ϕ and ψ are states, or formally, functionals J C, such that ϕ(a) and ψ(a) are the values of the observable a at those states respectively, and where is the operator norm, defined for a linear operator T acting between two normed vector spaces X and Y :

$$||T|| = \sup_{v \in X, v \neq 0} \frac{||T(v)||_Y}{||v||_X}$$

Where T (v) Y is the norm of the vector T (v) in the vector space Y , and v X is the norm of the vector v in the vector space X.

To justify the correctness of our assumption, we shall now verify the prop-erties that the metric gD must satisfy from the definition of a metric itself.

- Non-negativity: gD(ϕ, ψ) 0. This is readily true since the metric gD is the supremum of a modulus, which cannot be negative.
- Identity of indiscernibles: gD(ϕ, ϕ) = 0. Evident from definition itself.
- Converse of (2): gD(ϕ, ψ) = 0 implies ψ and ϕ are the same state. gD(ϕ, ψ) = 0 implies that ϕ(a) = ψ(a) for every observable a J . Two states ϕ and ψ are equal if and only if ϕ(a) = ψ(a) for all observables a ∈ AJ .
- Commutativity: gD(ϕ, ψ) = gD(ψ, ϕ). Evident from definition. |ϕ(a) − ψ(a)| = |ψ(a) − ϕ(a)|.
- The triangle inequality: For a third state ξ, the following must always hold: gD(ϕ, ψ) ≤ gD(ϕ, ξ) + gD(ψ, ξ). Proof: For the following triangle inequality holds for the modulus |ϕ(a)−ψ(a)| ≤ |ϕ(a)−ξ(a)|+|ψ(a)−ξ(a)|.

$$S = \{a \in \mathcal{A}_J : ||[\mathcal{D}_{DMB}, a]|| \leq 1\}$$

Then,

$$g_D(\phi, \psi) = \sup_{a \in S} |\phi(a) - \psi(a)|$$

Then, by definition of the supremum

$$|\phi(a) - \xi(a)| \leq \sup_{b \in S} |\phi(b) - \xi(b)| = g_D(\phi, \xi)$$

$$|\psi(a) - \xi(a)| \leq \sup_{b \in S} |\psi(b) - \xi(b)| = g_D(\psi, \xi)$$

Thus,
$|\phi(a)-\psi(a)| \le |\phi(a)-\xi(a)| + |\psi(a)-\xi(a)| \le g_D(\phi,\xi) + g_D(\psi,\xi)$

Define,

$$V = \{|\phi(a)-\psi(a)| : a \in S\}$$

Then gD(ϕ, ξ) +gD(ψ, ξ) is an upper bound of V . Since sup V is the lowest such upper bound, we have:

gD(ϕ, ψ) = sup |ϕ(a) − ψ(a)| = sup V ≤ gD(ϕ, ξ) + gD(ψ, ξ)

Thus, proved.
Thus gD satisfies the above five properties and can therefore be called a metric.

The Algebra
Define the algebra AF as:
AF = C ⊕ H ⊕ M3(C)
Where C is the algebra of all complex numbers, H is the algebra of the quater-nions, and M3(C) is the algebra of all 3 3 matrices whose entries are part of C. In the view of the standard model, C encodes the hypercharge, H the electroweak symmetry, and M3(C) the strong colour symmetry SU(3).
Define the algebra Aspacetime as:

$$A_{spacetime} = (C^{\infty}(M), \star_{\phi})$$

Where M is R4, or Minkowski space, equipped with a Minkowski metric η = diag(1, 1, 1, 1), and ⋆ϕ is the Moyal Product constrained by ϕ, where ϕ is the golden ratio defined earlier. This constrained Moyal Product is given by, for f, g ∈ Aspacetime:

$$f \star_{\phi} g(x) = f(x) \cdot \left[\sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{i}{2} \sum_{\mu=1}^{4} \sum_{\nu=1}^{4} \theta_{\phi}^{\mu\nu} \frac{\partial}{\partial x^{\mu}} \frac{\partial}{\partial y^{\nu}} \right)^{k} \right] g(y) \Big|_{x=y}$$

Or,

$$f \star_{\phi} g(x) = f(x) \cdot \exp\left[\left(\frac{i}{2} \sum_{\mu=1}^{4} \sum_{\nu=1}^{4} \theta_{\phi}^{\mu\nu} \frac{\partial}{\partial x^{\mu}} \frac{\partial}{\partial y^{\nu}} \right) \right] g(y) \Big|_{x=y}$$

Where θμν is the antisymmetric constant matrix constrained by ϕ, which satisfies:

$$\theta_{\phi}^{\mu\nu} = \phi^{-2} l_P^2 \theta_{canonical}^{\mu\nu}$$

Where $\theta_{canonical}^{\mu\nu}$ being skew symmetric, satisfies:

$$\theta_{canonical}^{\mu\nu} = -\theta_{canonical}^{\nu\mu}$$

(antisymmetry)

$$\theta_{canonical}^{\mu\mu} = 0$$

(entries on diagonals are zero) Define the Moyal product commutator:

$$[\mathbf{x}, \mathbf{y}]_{\star} = x \star y - y \star x$$

Then, for $\mu, \nu \in \{0,1,2,3\}$,

$$[\mathbf{x}^{\mu}, \mathbf{x}^{\nu}]_{\star_{\phi}} = i\theta_{\phi}^{\mu\nu} = i\phi^{-2} l_P^2 \theta_{canonical}^{\mu\nu}$$

Where l_P is the Planck length, and l_P^2 is the Planck area, and $\theta_{canonical}^{\mu\nu}$ is given by:

$$\theta_{canonical}^{\mu\nu} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

It can easily be verified that this choice of θμν canonical breaks Lorentz invariance. To define xμ, since this is an NCG, we need to have some considerations. For all vectors ξ, ψ in HD , and linear operator A acting on HD , define the Hermitian adjoint of A as the operator A† which satisfies

$$\langle \xi, A\psi \rangle = \langle A^{\dagger}\xi, \psi \rangle$$

where ⟨•, •⟩ is the inner product of the Hilbert space, defined as

$$\langle \xi, \psi \rangle = \sum \xi_i \bar{\psi_i}$$

If A = A†, then A is Hermitian. Thus, the coordinates xμ and xν , defined as operators R4 → R themselves must be Hermitian. Also, they generate the entire algebra Aspacetime, thus, for any smooth f(x) ∈ Aspacetime, we have:

$$f(x) = f(0) + \frac{\partial f}{\partial x^{\mu}}\Big|_{x=0} \star_{\phi} x^{\mu} + \frac{1}{2!} \frac{\partial^2 f}{\partial x^{\mu} \partial x^{\nu}}\Big|_{x=0} \star_{\phi} x^{\mu} \star_{\phi} x^{\nu} + \ldots$$

Additionally, they satisfy the following modified Heisenberg uncertainty principle:

$$\Delta x^{\mu} \Delta x^{\nu} \ge \frac{1}{2} |\theta_{\phi}^{\mu\nu}|$$

And, for f, g ∈ Aspacetime, we have:

$$[f,g]_{\star_\phi} = i\sum_{\mu=0}^{3}\sum_{\nu=0}^{3}\theta_\phi^{\mu\nu}\left(\frac{\partial f}{\partial x^\mu}\frac{\partial g}{\partial x^\nu} - \frac{\partial f}{\partial x^\nu}\frac{\partial g}{\partial x^\mu}\right) + i\mathcal{O}(\theta^3)$$

Where the minor correction terms O(θ3) are given by:

$$\mathcal{O}(\theta^3) = \sum_{\mu=0}^{3}\sum_{\nu=0}^{3}\sum_{\alpha=0}^{3}\sum_{\beta=0}^{3}\sum_{\gamma=0}^{3}\sum_{\delta=0}^{3}\frac{1}{240}\theta_\phi^{\mu\nu}\theta_\phi^{\alpha\beta}\theta_\phi^{\gamma\delta}\left(\frac{\partial^3 f}{\partial x^\mu\partial x^\alpha\partial x^\gamma}\frac{\partial^3 g}{\partial x^\nu\partial x^\beta\partial x^\delta} - \frac{\partial^3 g}{\partial x^\mu\partial x^\alpha\partial x^\gamma}\frac{\partial^3 f}{\partial x^\nu\partial x^\beta\partial x^\delta}\right)+\mathcal{O}(\theta^5)$$

Where the (θ5) is a negligible correction term of 10th order derivatives.

Then the algebra of the DMB space is given by:

$$\mathcal{A}_J = \mathcal{A}_{spacetime} \otimes \mathcal{A}_F$$

From which we define multiplication for elements of AJ . For f, g ∈ Aspacetime and a1, a2 ∈ AF , define:

$$(f \otimes a_1)\cdot(g \otimes a_2) = (f \star_\phi g) \otimes (a_1 a_2)$$

Where multiplication for a1 = (z1, h1, m1) and a2 = (z2, h2, m2) where z1, z2 ∈ C, h1, h2 ∈ H and m1, m2 ∈ M3(C) is defined as:

$$\mathcal{A}_J = \mathcal{A}_{spacetime} \otimes \mathcal{A}_F$$

From which we define multiplication for elements of AJ . For f, g ∈ Aspace time and a1, a2 ∈ AF , define:

$$(f \otimes a_1)\cdot(g \otimes a_2) = (f \star_\phi g) \otimes (a_1 a_2)$$

Where multiplication for a1 = (z1, h1, m1) and a2 = (z2, h2, m2) where z1, z2 ∈ C, h1, h2 ∈ H and m1, m2 ∈ M3(C) is defined as:

$$a_1 a_2 = (z_1 z_2, h_1 h_2, m_1 m_2)$$

Denote this value by a3. By the definition of an algebra, if M, N ∈ A then MN ∈ A. Thus, (f ⋆ϕ g) ⊗ a3 must be of the form f ⊗ a, where f ∈ A space time.

Thus,

$$(f \star_\phi g) \in (C^\infty(M), \star_\phi)$$

The Hilbert space

Next is the Hilbert space HD, defined as:

$$\mathcal{H}_D = L^2(M,S) \otimes \mathcal{H}_F$$

where M is R4, or the Minkowski space, equipped with a Minkowski metric η = diag(1, 1, 1, 1), and L2(M, S) is the space of all possible wave functions ψ(x) of a relativistic particle such that ψ(x) has four components, because S is the spinor bundle over the manifold M, ψ(x) is defined over Minkowski 4D spacetime, because the domain is M, and the probability density ψ2 is normalizable, because all L2 functions are square integrable, thus:

$$\int_M |\psi(x)^2| d\mu < \infty$$

Where the measure dμ = | det(gμν)|dx0dx1dx2dx3. Any state Ψ ∈ HD is of the form ψC ⊗ ψF where ψC ∈ L2(M, S), and ψF ∈ HF . For an observable A = f α J , where f spacetime and α F , we have, the action of the observable A onto the state Ψ is given by:

$$A \cdot \Psi = (f \star_\phi \psi_C) \otimes (\alpha \cdot \psi_F)$$

Where the second multiplication is standard matrix/vector multiplication. De-fine the inner product ⟨•, •⟩H on the Hilbert space HD for Ψ = ΨC ⊗ ΨF ∈ HD, and Φ = ΦC ⊗ ΦF ∈ HD, and ΨC, ΦC ∈ L2(M, S), ΨF , ΦF ∈ HF , we have:

$$\langle \Psi, \Phi \rangle_D = \langle \Psi_C, \Phi_C \rangle_C \cdot \langle \Psi_F, \Phi_F \rangle_F$$

$$\langle \Psi, \Phi \rangle_D = \overline{\langle \Phi, \Psi \rangle_D}$$

Where ⟨•, •⟩F is the inner product of a finite Hilbert space, given by:

$$\langle \xi, \psi \rangle = \sum \xi_i \bar{\psi_i}$$

And ⟨•, •⟩C is the inner product over L2(M, S), given by:

$$\langle \psi, \xi \rangle_C = \int_M \psi^\dagger(x) \cdot \gamma^0 \cdot \xi(x) d\mu$$

Where the Hermitian adjoint, ψ† is defined as:

$$\psi^\dagger = (\psi^{\mathbf{T}})^*$$

Where the operation ( )∗ refers to complex conjugating every element of ψ, and the Dirac matrix γ0 is defined as:

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Thus, define a state or a linear functional χ : AJ → C, or a vector state or a pure state associated with Ψ D. Then, χ(A), i.e. the expectation value of observable A, is equivalent to:

$$\chi(A) = \langle \Psi, A\Psi \rangle_D$$

Additionally, there exists a grading operator Γ acting on HD, given by:

$$\Gamma = \gamma^5 \otimes 1_F$$

Where 1F is the identity matrix in F such that for any Ψ F , 1F Ψ = Ψ, and γ5 is the fifth traceless Dirac matrix:

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Γ is Hermitian: Γ = Γ†, and for any Ψ ∈ HD, Γ2Ψ = Ψ. It anti-commutes with the Dirac operator DDMB:

$$\{\Gamma, \mathcal{D}_{DMB}\} = 0$$

There is also a real structure operator J on D. For Ψ, ξ D and αC, we have:

$$J(c\Psi) = \bar{c}J(\Psi)$$

$$\langle J\Psi, J\xi \rangle_D = \langle \xi, \Psi \rangle_D = \overline{\langle \Psi, \xi \rangle_D}$$

$$J^2\Psi = \Psi$$

Depending on the KO-dimension, here taken 6 (mod 8) of the real spectral triple we impose either [J, D] = 0 or J, D = 0 as appropriate. In this paper we assume the KO-dimension for which [J, D] = 0. The commutativity of J with DMB establishes that the mass spectrum, or the eigenvalues of DMB, is same for a particle and its antiparticle, establishing Charge-parity symmetry in the DMB geometry.

The Dirac operator

Next is the Dirac operator DDMB, a non-commutative tensor product Hermitian operator acting on HD, expressed as:

$$\mathcal{D}_{DMB} = \mathcal{D}_{frac} \otimes \mathbf{1}_F + \gamma_5 \otimes D_F$$

Also:

$$\mathcal{D}_F = \mathcal{D}_F^\dagger$$

Where $\mathcal{D}_{frac}$ is defined as:

$$\mathcal{D}_{frac} \sim (-\Delta)^{D_{DMB}/2}$$

Here denotes equality of principal symbols, i.e., we identify the leading-order pseudo-differential behaviour of frac with that of (Δ)s. Here, Δ is the D'Alembertian operator:

$$\Delta = \partial^\mu \partial_\mu = \sum_{\mu=0}^{3}\sum_{\nu=0}^{3} \eta^{\mu\nu} \frac{\partial^2}{\partial x^\mu \partial x^\nu}$$

Where $\eta^{\mu\nu}$ is given by:

$$\eta^{\mu\nu} = \text{diag}(1, -1, -1, -1)$$

Or in general,

$$\eta^{\mu\nu}\eta_{\nu\lambda} = \delta^\mu_\lambda$$

Where δμ is the Kronecker delta. To get an idea of how the structure of Dfrac is, consider a state Φ D, and its Fourier transform into momentum space Φ˜. Then:

$$\Phi(x) = \int \frac{d^4p}{(2\pi)^4} e^{-ipx}\tilde{\Phi}(p)$$

Where $px = \sum\sum \eta_{\mu\nu} p^\mu x^\nu$. To both sides apply the D'Alembertian operator:

$$\Delta\Phi(x) = \Delta\left(\int \frac{d^4p}{(2\pi)^4} e^{-ipx}\tilde{\Phi}(p)\right) = \sum\sum \eta^{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu}\int \frac{d^4p}{(2\pi)^4} e^{-ipx}\tilde{\Phi}(p)$$

$$= \int \frac{d^4p}{(2\pi)^4}\tilde{\Phi}(p)\sum\sum \eta^{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu} e^{-ipx}$$

$$= \int \frac{d^4p}{(2\pi)^4}\tilde{\Phi}(p)\sum\sum \eta^{\mu\nu}\frac{\partial}{\partial x^\mu}(-ip_\mu e^{-ipx})$$

$$= \int \frac{d^4p}{(2\pi)^4}\tilde{\Phi}(p)\sum\sum \eta^{\mu\nu}(-1)p_\mu p_\nu e^{-ipx}$$

$$= \int \frac{d^4p}{(2\pi)^4}\tilde{\Phi}(p)e^{-ipx}\sum\left(\sum -\eta^{\mu\nu}p_\mu p_\nu\right)$$

$$= \int \frac{d^4p}{(2\pi)^4}\tilde{\Phi}(p)e^{-ipx}\sum\left(\sum -p^\nu p_\nu\right)$$

$$= \int \frac{d^4p}{(2\pi)^4}\tilde{\Phi}(p)e^{-ipx}(-\mathbf{p}^2)$$

Where p2 = (p0)2 (p1)2 (p2)2 (p3)2 is the squared four-momentum. Define Ψ = ΔΦ. Then, from the definition of the Fourier transform:

$$\Psi(x) = \int \frac{d^4p}{(2\pi)^4} e^{-ipx}\tilde{\Psi}(p)$$

But:

$$\Psi(x) = \int \frac{d^4p}{(2\pi)^4}\tilde{\Phi}(p)e^{-ipx}(-\mathbf{p}^2)$$

Thus,

$$\tilde{\Psi}(p) = (-\mathbf{p}^2)\tilde{\Phi}(p)$$

Thus the Fourier transform of Ψ = ΔΦ is (−p2)Φ. Thus the momentum space representation L(p) of the D'Alembertian operator is L(p) = −p2. For a state Ψ ∈ HD, define the fractional D'Alembertian operator Δs as:

$$\mathcal{F}(\Delta^s\Psi(x)) = (-\mathbf{p}^2)^s\tilde{\Psi}(p)$$

Where F represents the Fourier transform. Take the inverse Fourier:

$$\Delta^s\Psi(x) = \int \frac{d^4p}{(2\pi)^4} e^{-ipx}\tilde{\Psi}(p)(-\mathbf{p}^2)^s$$

Here (p2)s denotes the fractional power of the Lorentzian invariant. We inter-pret fractional powers via spectral calculus on a self-adjoint Euclidean Dirac D, equivalently using the Euclidean momentum symbol |pE|2s rather than (p2)s.

p2 = (p0)2 p⃗ 2, interpreted using the principal branch of the complex power function. It is easily shown that:

$$\Delta \xrightarrow{\mathcal{F}} -p^2 \implies -\Delta \xrightarrow{\mathcal{F}} p^2$$

$$\implies \mathcal{D}_{frac}\Psi(x) \sim \int \frac{d^4p}{(2\pi)^4} e^{-ipx}\tilde{\Psi}(p)(\mathbf{p}^2)^s$$

To get an exact definition for Dfrac, set s = DDMB/2. Let D be a Hermitian Dirac operator acting on HD with spectral decomposition:

$$Du_k = \lambda_k u_k$$

Where u_k is an orthonormal basis of eigenvectors with real eigenvalues λ_k.

$$\langle u_i, u_j\rangle_D = \delta^i_j$$

Such that all $\Psi \in \mathcal{H}_D$ can be expressed, for some $c_i \in \mathbb{C}$:

$$\Psi = \sum c_i u_i$$

And the action of D on Ψ is given by:

$$D\Psi = \sum \lambda_i c_i u_i$$

Thus for any function f,

$$f(D)u_k = f(\lambda_k)u_k$$

Let $\mathcal{D}_{frac} = D|D|^{s-1}$. Then, we have, for $f(X) = |X|$:

$$|D|u_k = |\lambda_k|u_k$$

For $f(X) = |X|^{s-1}$:

$$|D|^{s-1}u_k = |\lambda_k|^{s-1}u_k$$

Also observe that for any real α:

$$\alpha|\alpha|^{p-1} = |\alpha|^p\,\mathrm{sgn}(\alpha)$$

Thus:

$$\mathcal{D}_{frac}u_k = D|D|^{s-1}u_k = \lambda_k|\lambda_k|^{s-1}u_k = |\lambda_k|^s\,\mathrm{sgn}(\lambda_k)u_k$$

For the purposes of spectral calculus and heat-kernel asymptotics below we work on the Euclideanized manifold, or the Wick-rotated metric so that the Dirac operator is elliptic and self-adjoint. After Wick rotation the operator Δ is understood as the positive Euclidean Laplacian ∂i∂i, whose symbol is pE 2. Lorentzian signature expressions appearing above are to be understood in the analytic continuation sense.

Representation of the algebra

Define an operator T on the Hilbert space HD to be bounded if there exists a constant C such that for all vectors ψ ∈ HD,

$$||T\psi|| \leq C||\psi||$$

Define the set of all bounded operators on HD as B(HD). It is a C∗ algebra. Define the representation π of the algebra AJ as the linear map π : AJ → B(HD) such that π satisfies, for abstract observables A, B in AJ , and α, β ∈ C,

$$\pi(AB) = \pi(A)\pi(B)$$

$$\pi(A) = \pi(A)^\dagger$$
$$\pi(\alpha A + \beta B) = \alpha\pi(A) + \beta\pi(B)$$
$$\pi([A, B]) = [\pi(A), \pi(B)]$$
$$\langle\pi(A)\psi, \pi(B)\psi\rangle_D = \langle\pi(B)\psi, \pi(A)\psi\rangle_D$$
$$\pi(1_A) = 1_H$$

Where 1A is the identity element of AJ and 1H is the identity element of HD. Define the physical operator Aˆ = π(A) where A ∈ AJ is an observable, such that for any vector ψ D,

$$\hat{A}\psi = \pi(A)\psi$$

Since AJ is assumed to be represented on HD as a C∗ algebra, there exists an operator ∗ such that, for all observables x, y in AJ and for all λ ∈ C:

$$(x^*)^* = x$$
$$(x + y)^* = x^* + y^*$$
$$(xy)^* = y^*x^*$$
$$(\lambda x)^* = \bar{\lambda}x^*$$
$$x = x^*$$

The last equality arising from the algebraic version of the Hermitian. Addition-ally, for symbols as defined above:

$$\pi(f \otimes a)\,(\Psi_C \otimes \Psi_F) = (f \star_\phi \Psi_C) \otimes (a\Psi_F)$$

These properties of π are essential for defining a consistent framework for quan-tum mechanics within the DMB geometry. In particular, the representation preserves the algebraic structure of J while ensuring that observables repre-sented by Hermitian operators act on the state vectors in a way that is con-sistent with quantum mechanical principles like measurement, uncertainty, and time evolution

An Illustration in the Semi-classical Limit

Define DF as the Dirac operator on the algebra AF , given by:

$$D_F = \sum_{\mu=0}^{3} \gamma^\mu(\partial_\mu + A_\mu)$$

Where γ0 is as above, and the rest of the Dirac matrices are given by:

$$\gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

And define the operator ∂μ as:

$$\partial_\mu = \frac{\partial}{\partial x^\mu}$$

Aμ represents the gauge field or the connection for the interaction of a fermion with the gauge field. As a side note, observe that AF represents a 'fermionic' algebra, and the Dirac operator DF a 'fermionic' Dirac operator. For the elec-tromagnetic force, Aμ is associated with the photon, which is the mediator of the electromagnetic force. It is then defined as the four-potential, a vector, and it takes the form:

$$A_\mu = (A_0, \mathbf{A})$$

Where A0 is the scalar potential and A is the vector potential. Given electric field E and magnetic field B acting on the fermion, the following hold:

$$\mathcal{B} = \nabla \times \mathbf{A}$$

$$\mathcal{E} = -\nabla A_0 - \frac{\partial \mathbf{A}}{\partial t}$$

Where ∇A0 is the gradient of A0 given by:

$$\nabla A_0 = \sum \frac{\partial A_0}{\partial x^\mu} e_\mu$$

Where eμ are coordinate basis vectors, and the curl ∇ × A is defined as:

$$\nabla \times \mathbf{A} = \sum_\mu \left(\sum_\nu \sum_\alpha \varepsilon^{\mu\nu\alpha} \frac{\partial A^\alpha}{\partial x^\nu} \right) e_\mu$$

Where εμνα is the Levi-Civita symbol, given by:

$$\varepsilon^{\mu\nu\alpha} = \begin{cases} 1 & \text{if } (\mu,\nu,\alpha) \text{ is an even permutation of } (1,2,3) \\ -1 & \text{if } (\mu,\nu,\alpha) \text{ is an odd permutation of } (1,2,3) \\ 0 & \text{if any two indices are the same} \end{cases}$$

For QCD, the gauge field Aμ is written as:

$$A_\mu = A_\mu^\nu T^\nu$$

Where Aν are components of Aμ, and T ν are matrices that represent the gen-erators of the SU(3)C group of colour symmetry, given by:

$$T^\nu = \frac{1}{2}\vartheta^\nu$$

Where ϑν are the Gell-Mann matrices, a set of eight 3x3 linearly independent traceless Hermitian matrices, given by:

$$\vartheta_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\vartheta_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\vartheta_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\vartheta_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\vartheta_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}$$

$$\vartheta_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\vartheta_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$\vartheta_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

III. QUANTUM MECHANICS ON THE DMB GEOMETRY

We now formulate quantum mechanics on the Daksh–Majumder–Balaji geom-etry. Since the configuration space of the theory is encoded by the spectral triple the algebra of observables, the state space, and the time evolution must be expressed in this framework. The formulation given below is a geometric refinement of the Dirac–von Neumann axioms of quantum theory.

1. State Space

At a fixed physical time parameter τ , the state of an isolated system is repre-sented by a unit vector

$$|\psi\rangle \in \mathcal{H}_D, \qquad \|\psi\| = 1.$$

Two vectors differing by a global phase represent the same physical state:

$$|\psi\rangle \sim |\phi\rangle \quad \Longrightarrow \quad |\psi\rangle = e^{i\alpha}|\phi\rangle, \ \alpha \in \mathbb{R}.$$

Thus the physical pure states correspond to rays in HD, i.e. to the projective space P(HD).

In addition, every unit vector |ψ⟩ defines an algebraic state on AJ ,

$$\omega_\psi(a) = \langle \psi, \pi(a)\psi \rangle_D,$$

where π is the representation of AJ .

Measurement Postulate

For every physical quantity P there exists an observable Pˆ = Pˆ† = π(P) ∈ π(J). The possible results of measuring in the state ψ are the eigenvalues of Pˆ. Immediately after the measurement yielding eigenvalue λ, the state collapses to the corresponding normalized eigenvector.

Dynamics

The physical (Lorentzian) time evolution is governed by the Schr¨odinger equa-tion,

$$i\hbar \frac{\partial}{\partial \tau} |\psi(\tau)\rangle = \hat{H}(\tau)\, |\psi(\tau)\rangle,$$

where Hˆ(τ) = Hˆ(τ)† ∈ π(AJ) is the Hamiltonian. This real-time evolution is distinct from the Euclidean geometric framework used in the definition of J . The two are related by analytic continuation when necessary.

Composition of Identical Particles

For a system of N identical particles, the total state belongs to the tensor prod-uct ⊗N and is either totally symmetric (for bosons) or totally antisymmetric (for fermions) under interchange of any two particle labels. The symmetrization or antisymmetrization acts on both the spacetime spinor factor and the internal degrees of freedom appropriate to the particle species.

Expectation Values

If Aˆ = Aˆ† π(J) is an observable and ψ D is a state, the expectation value is

$$\langle \hat{A} \rangle_\psi = \frac{\langle \psi, \hat{A}\psi \rangle_D}{\langle \psi, \psi \rangle_D}.$$

This formulation also induces algebraic states ωψ(a) = ψ, π(a)ψ D, which are used in the Connes' spectral distance formula provided earlier.

Quantum Field Theory on the DMB geometry

The Spectral Action of the DMB geometry

In the NCG framework assumed in the paper, the dynamics and state infor-mation of the physical system under consideration are encoded in the spectra of the Dirac operator DMB. For the Spectral Triple of the DMB geometry, the bosonic Lagrangian is obtained from the Spectral Action principle acting on DDMB:

$$\mathcal{S}[\mathcal{D}^2_{DMB}] = \mathrm{Tr}\left(f\left(\frac{\mathcal{D}_{DMB}}{\Lambda} \right) \right)$$

Where Tr is the trace of an operator, defined for an operator A as:

$$\mathrm{Tr}(A) = \sum \langle u_i, T u_i \rangle$$

Where ui is an orthonormal basis:

$$\langle u_i, u_j \rangle = \delta^i_j$$

Represent the eigenvalues of DDMB by λi. These are all real, by virtue of the fact that DDMB is Hermitian. Then,

$$\mathrm{Tr}\left(f\left(\frac{\mathcal{D}_{DMB}}{\Lambda} \right) \right) = \sum f\left(\frac{\lambda_i}{\Lambda} \right)$$

Such that f is a positive cut-off function, thus, f ∈ C∞, it is close to 1 for small eigenvalues, and quickly becomes zero for large eigenvalues, and f(λi)   0. This is required since of the infinite eigenvalues of the Dirac operator, only a finite amount are needed for the Spectral Action. Λ is the unification scale as determined by Connes-Chamseddine ≈ 1016GeV. Effectively, high-energy modes, i.e. λi   Λ are 'suppressed'. The asymptotic expansion of is given by:

$$S \sim \sum_{k \geq 0} f_k \, a_k(\mathcal{D}^2_{DMB}) \, \Lambda^{D_{DMB}-k}$$

Where ai are the generalized Seeley-DeWitt coefficients, and fk are the moments of the cut-off function f, defined as:

$$f_k = \int_0^{\infty} f(u) u^{k-1} du$$

$$\text{for } k > 0, \text{ and } f_0 = f(0)$$

IV. CONCLUSION AND FUTURE WORK

We have proposed a new Planck-scale geometry described by the DMB spectral triple, characterized by a Golden-ratio spectral dimension, a ϕ−2 scaled non-commutativity, and a fractional Dirac operator. This preserves the algebraic framework for quantum states and observables, while introducing modified ul-traviolet behaviour through DDMB.

Future work will focus on computing heat-kernel coefficients ak(D2) to extract the gravitational and gauge dynamics from the spectral action, as well as exploring phenomenological consequences and possible experimental tests.

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