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Time-Dependent Comprative Analysis of Analytical and Numerical Solution for First-Order Ordinary Differential Equations

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Abstract- This paper presents a time-dependent comparative study between numerical and analytical solutions of first-order ordinary differential equations (ODEs), with a focus on accuracy and computational performance. Numerical methods such as Taylor Method, and the classical fourth-order Runge-Kutta method are implemented and analyzed using PYTHON. The study investigates how the accuracy of each numerical method varies with time when compared to the exact analytical solution. To demonstrate this, Newton's Law of Cooling is used as a case study, providing a real-world application where temperature changes over time are modeled by a first-order ODE. Graphical comparisons and error analyses are carried out over different time intervals to evaluate the stability and convergence behavior of each numerical method. The results reveal that while all methods can approximate the solution, the Runge-Kutta method offers the best balance of accuracy and computational efficiency. This work highlights the importance of method selection in numerical analysis and the effectiveness of PYTHON in simulating and visualizing dynamic systems.

Keywords: Python, Time-Dependent Analysis; Numerical Methods; Error Analysis; Scientific Computing, etc.

I. INTRODUCTION

Differential equations have the fundamental importance in different fields of science, i.e. applied sciences (Engineering, physics, biology, Economics and Finance, Climate Science) as many physical laws and relation, whom we define mathematical formulation in the form of differential equations .The differential equations have many solutions in the daily life, it leads us to numerous solutions of physical and geometrical problems stated that many. Many numerical techniques can applied to acquire an approximation to the solution of differential equations. The literature review discloses that a significant number of authors have employed several methods for numerical solutions.

Many researchers have tried to solve the initial value problem (IVP) to obtain a more accurate solution based on the Taylor method and RK4 method etc. Moreover, the real life mathematical problems can be solved by forming the differential equations. In real conditions, differential equations are difficult to solve exactly. Computer simulations and numerical approximations are beneficial for the solution of the complex systems describing that many real-life processes models with a system of differential equations. Even though the computation of higher order derivatives is complicated, this method can give reliable starting values and is particularly suitable also. When the solution to ODEs displays swift variation over a part of time interval but slower variation over the rest of the time interval, applying a fixed time steps is inexpedient. In numerical analysis, the adjustable step size method has ordinarily been used to control the method's errors and to ensure stability properties.

The still better approach to suitable results known two of methods is Taylor method and RK4 method. Three examples of different kinds of ODEs are given for verifying the proposed formulation. The outcomes of numerical example show that the approach's effectiveness is demonstrate by the provided convergence and error analysis [2]. While the normal behaviors of convergence theory by Taylor method and RK4 method can be simply adapted to include these two methods, there is some improvement in having a theory which includes them in a completely natural way [4]. In direction to achieve better accuracy in the solution, the step size wants to be very small. Finally we examine and compute the errors of the two suggested methods for dissimilar step sizes to examine advantage. Numerous numerical examples are given to determine the consistency and proficiency [5].

Researchers discussed on accurate solution of initial value problems for ordinary differential equations, Measured on some numerical methods for solving initial value problems in ordinary differential equations, also considered numerical solutions of initial value problems for ordinary differential equations using a multiplicity of numerical methods [6]. In this work Taylor method and RK4 methods are performed without any classification, modification or restraining hypothesis for solving initial value problems of ordinary differential equations, On the diverge, RK4 method provides enhanced results and it converges closer to analytical solution and has less iteration to get meticulousness solution.

II. APPLICATIONS FOR THE FIRST ORDER ORDINARY

Differential Equations to heat Convection in Fluid.

When heat is transmitting the technique of transmission of thermal strength in the bodily gadget relies upon the pressure and temperature, via dispersing heat is describing by using heat remodeling, heat commonly is transforming through transmission or dispersion, convection and emission. Hotness usually transmitted by way of one of the following.

- Heat transferring by way of transmission to solid;
- Conveying heat by way of convection in the fluid;
- Conveying heat by using radiation in space;

therefore, each time heat moves within the strong by conduction those can be resolute by way of Fourier regulation inside the solid. We recognize that hotness always waft from excessive to low temperature with the aid of nature. Whereas in fluid heat is flown by means of convection which will be calculated by the Newton's Law of Cooling.

Mathematical Model of Heat Transformation in Fluid

Due to the fact that heat glide within fluid through convection "as proven in figure 2" [1], So we are able to determine, given below equation no one which is identified as the Newton's law of Cooling.

$$q\beta(T_a - T_b) = h(T_a - T_b) \qquad (1)$$

The above Equation display that heat flows from to with > which indicates the heat flux among two points A as surface area and h is the coefficient of heat transfer.

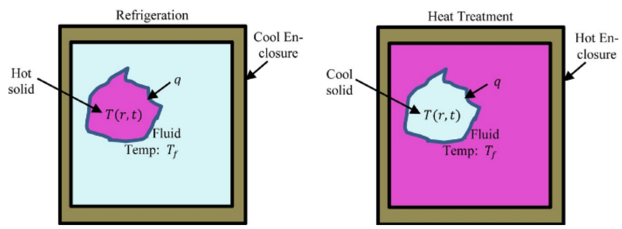

Figure1. Heat transmitting solid immersed in fluid. [1]

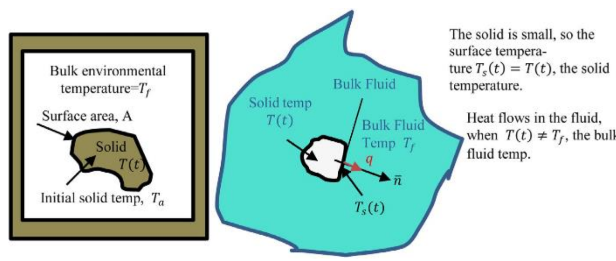

Figure2. Mathematical model of heat transmission, [1]

• Summarize existing research relevant to our topic.

$$q = h[T_s - T_f] = h[T - T_f] \quad . \qquad (2)$$

Inside the above equation (2) the coefficient h is a variety of that satisfies the heat conversion among solid and bulk fluid. Likewise using first law of thermodynamics, to supplying temperature alternate in a solid throughout the term , may be achieved by the principle.

$$-\rho cV\delta T = Q = qA_s\delta t = hA_s[T - T_f]\delta T \qquad (3)$$

From Equation (3), we have

$$\frac{\delta T(t)}{\delta t} = -\frac{h}{\rho c V}A_s\left[\mathrm{T} - T_f\right] \qquad (4)$$

From the given Equation (4) ρ, h, c, and V from right-hand-side are constants by comparing given constants with β.

$$\beta = \frac{h}{\rho c V} \quad (5)$$

Further Equation (4) can be expressed as:

$$\frac{\delta T(t)}{\delta t} = -\beta A_s[T - T_f]. \quad (6)$$

Hence the job of temperature of immersed solid T(t) is continuously recognized within time t seconds, i.e. $\delta t \rightarrow 0$ and we change the steady floor location to an ordinary image. As we express the Equation within shape of first-order Ordinary Differential Equation (6).

$$\frac{dT(t)}{dt} = -\beta A_s[T - T_f]. \quad (7)$$

With Initial Condition (IC)

$$T(t) = 0 = T(0) = T_0 .$$

Furthermore, Taylor technique is not often used, but this approach have to be understood a good way to understand the RK4 method. If x_i and y_i are known exactly, then Taylor collection growth is

$$y(x_i + 1) = y(x_i + h) = y(x) + hy'(x) + \frac{h^2}{2!}y''(xi) + \frac{h^3}{3!}y'''(xi) + \frac{h^4}{4!}y''''(x_i) + \cdots . + \frac{h^n}{n!}y^n(x_i) + \frac{h^{n+1}}{(n+1)!}y^{(n+1)}(x_i \quad (8)$$

III. NUMERICAL METHOD OF FIRST ORDER ORDINARY DIFFERENTIAL EQUATIONS

Herein phase, we discuss two numerical samples and decide one of the discussed numerical methods convergence extra quick toward the exact results. We will calculate results numerically and check errors and show them graphically to explain our findings.

Method of Taylor for Solution of the (ODEs)

Taylor approach is numerically system is used to solve first order ordinary differential equations, through approximating solutions via the enlargement the Taylor series. This technique is in general useful when solving preliminary cost troubles (IVPs) of the form: $\mathbf{y'(x) = f(x, y)}$, $\mathbf{y(x_0) = y_0}$ (09)

Well known concept

The Taylor approach approximates the answer by way of increasing the answer characteristic y(x) as a Taylor collection method about a factor (point) . For a given width or step size h, the Taylor collection growth of the answer across the point is:

$$\mathbf{y(x_0 + h) = y(x_0) + hy'(x_0) + \frac{h^2}{2!}y''(x_0) + \frac{h^3}{3!}y'''(x_0) + \cdots \ldots \ldots} \quad (10)$$

Truncating this collection to a certain wide variety of terms offers an approximation of y(x) at a new factor.

Taylor series Approximation

In exercise, the Taylor method may be defined for different orders depending on how many terms of the collection we hold. For instance:

- .**Method of Taylor for First-order**: This is simply the Euler method.
  $$y_{n+1} = y_0 + hf(x_n, y_n) \quad (11)$$
- **Taylor Method of Second-order**: The series is truncated after the second term. $y_{n+1} = y_n + hf(x_n, y_n) + \frac{h^2}{2!}f_x(x_n, y_n)$ (12)
- **Taylor Method of Third-order**: The series is truncated after third term.

$$y_{n+1} = y_n + hf(x_n, y_n) + \frac{h^2}{2!}f_x(x_n, y_n) + \frac{h^3}{3!}f_{xx}(x_n, y_n) \quad (13)$$

Where:

- f_x and f_{xx} are derivatives of $f(x, y)$ partially with respect to x;
- f_y and f_{yy} are derivatives of $f(x, y)$ partially with respect to y;
- Higher-order terms involve mixed partial derivatives, i-e f_{xy} and f_{yx};

Solving ODEs Using the Taylor Method

1. **Initial Value**: Start with the initial conditions x_0 and y_0.
2. **Choose Step Size** h: Select a step size for the approximation.
3. **Compute Derivatives**: For each step, calculate the compulsory derivatives of $f(x, y)$ at every point.
4. **Iterate**. We use the Taylor series expansion for evaluation the value of y at subsequent points.

Example: 1. **Let's solve a simple ODE using the 2nd-order Taylor method:**

$$y'(x) = -2xy, \quad y(0) = 1 \quad (14)$$

- Compute $f(x, y) = -2xy$.
- The first derivative of function f with respect to x is $\frac{\partial f}{\partial x} = -2y$, $\frac{\partial f}{\partial y} = -2x$

- The second derivatives are $\frac{\partial^2 f}{\partial x^2} = 0$, $\frac{\partial^2 f}{\partial y^2} = 0$, $f_{xy} = -2$ *and* $f_{yx} = -2$.

The second-order Taylor method then becomes: $y_{n+1} = y_n + hf(-2x, y_n) + \frac{h^2}{2!} f_x(-2x, y_n)$ (15)

Runge-Kutta of 4th Order (RK4) for the solution of First Order Ordinary Differential Equations (ODEs)

RK4 method is a generally used numerical technique for solving of ODEs. It's highly perfect and commonly applied in both theoretical and practical problems. Here's how the RK4 technique is typically applied to solve a 1st-order ODE:

Given Differential Equation is

$$\mathbf{y'(x) = f(x, y)} \qquad \mathbf{y(x_0) = y_0} \quad \textbf{(16)}$$

The RK4 Method Formula

The RK4 method provides an iterative approach to approximate the value of y(x) at discrete point.it uses four "slopes" to estimate the next value of y . For each step, you calculate:

1. *First slope* (k_1): $k_1 = hf(x_n, y_n)$

2. *Second slope* (k_2): $k_2 = hf(x_n + \frac{h}{2}, y_n + \frac{k_1}{2})$

3. *Third slope* (k_3): $k_3 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right)$

4. *Fourth slope* (k_4): $k_4 = hf(x_n + h, y_n + k_3)$

So we have y_{n+1}

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

Where h is the step size, and $x_{n+1} = x_n + h$.(18)

Steps for Implementation**:**

1. **Initialize** the values for x_0, y_0 , and the step size h.
2. **Iterate** through the above steps until you reach the desired value of x.
3. The values of y will be approximated by solution of differential equation with corresponding values of x.

Example: 2. Let solve the following first-order differential equation using RK4:

$$\frac{dy}{dx} = -2xy, \qquad y(0) = 1$$

1. **Define function**: $f(x, y) = -2xy$
2. **Initial condition**: $y(0) = 1$
3. **Choice of a step size** $(for\ example,\ h = 0.1\ or\ 0.5)$

Problem.1: Let 800C is the initial temperature of an object, if it is kept in a refrigerator, and preserve it at 50C. If and . Compute the temperature of an object at given time t seconds. Through Analytical solution and via numerically (Taylor and RK4 methods) and compare the results and errors of every approach up to 3000 seconds. And check the accuracy of methods near Analytical.

Solution:

Compare the given problem separately and combine with discussed methods

(a) The comparison between the analytical solution with Taylor method and RK4 for the temperature of the object over time is shown below, at step size 100s.

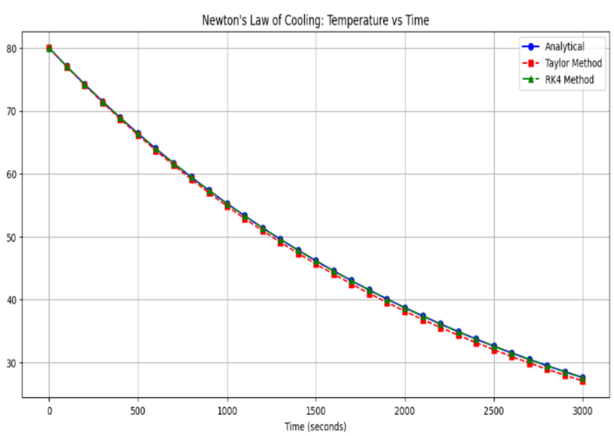

Chart -1

Table -1

T (s)	Exact	Taylor	RK4 (°C)	Error Taylor	Abserr_RK4 (°C)
0	80.00000	80.00000	80.00000	0.00000	0.00000
100	77.05921	77.06000	77.05921	0.00079	0.00000
200	74.23373	74.23525	74.23373	0.00152	0.00000

300	71.51903	71.52123	71.51903	0.00219	0.00000
400	68.91078	68.91359	68.91078	0.00281	0.00000
500	66.40481	66 .40818	66.40481	0.00337	0.00000
600	63.99709	64.00098	63.99709	0.00389	0.00000
700	61.68378	61.68814	61.68378	0.00436	0.00000
800	59.46118	59.46597	59.46118	0.00478	0.00000
900	57.32572	57.33090	57.32572	0.00517	0.00000
1000	55.274	55.27953	55.274	0.00552	0.00000
1100	53.30273	53.30857	53.30273	0.00584	0.00000
1200	51.40875	51.41488	51.40875	0.00612	0.00000
1300	49.58904	49.59541	49.58904	0.00637	0.00000
1400	47.84068	47.84727	47.84068	0.00659	0.00000
1500	46.16087	46.16766	46.16087	0.00678	0.00000
1600	44.54693	44.55389	44.54693	0.00695	0.00000
1700	42.99627	43.00338	42.99627	0.00710	0.00000
1800	41.50642	41.51364	41.50642	0.00722	0.00000
1900	40.07498	40.08231	40.07498	0.00732	0.00000
2000	38.69967	38.70708	38.69967	0.00740	0.00000
2100	37.37829	37.38576	37.37829	0.007475	0.00000
2200	36.10872	36.11624	36.10872	0.00752	0.00000
2300	34.88893	34.89649	34.88893	0.00755	0.00000
2400	33.71697	33.72454	33.71697	0.00757	0.00000
2500	32.59096	32.59854	32.59096	0.007583	0.00000
2600	31.5091	31.51668	31.5091	0.007577	0.00000
2700	30.46966	30.47722	30.46967	0.00756	0.00000

(b) Here is the comparison table with chart for Analytical, Taylor, and RK4 methods with their respective errors. The errors are calculated as the absolute difference between the numerical methods and the analytical solution with step size 180 seconds at the time interval 0 to 3100 seconds.

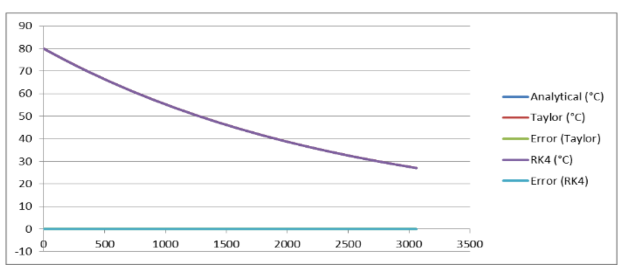

Chart:2

Table: 2

Time (s)	Analytical (°C)	Taylor (°C)	Error (Taylor)	RK4 (°C)	Error (RK4)
0	80.0000	80.0000	0.0000	80.0000	0.0000
180	74.7898	74.7944	0.0046	74.7898	0.0000
360	69.9416	69.9501	0.0085	69.9416	0.0000
540	65.4301	65.4421	0.0120	65.4302	0.0001
720	61.2321	61.2469	0.0148	61.2321	0.0000
900	57.3257	57.3429	0.0172	57.3257	0.0000
1080	53.6907	53.7099	0.0192	53.6907	0.0000
1260	50.3082	50.329	0.0208	50.3082	0.0000
1440	47.1607	47.1828	0.0221	47.1607	0.0000
1620	44.2318	44.255	0.0232	44.2318	0.0000
1800	41.5064	41.5304	0.024	41.5064	0.0000
1980	38.9704	38.9949	0.0245	38.9704	0.0000
2160	36.6105	36.6354	0.0249	36.6105	0.0000
2340	34.4145	34.4396	0.0251	34.4145	0.0000
2520	32.3711	32.3963	0.0252	32.3711	0.0000
2700	30.4697	30.4948	0.0251	30.4697	0.0000
2880	28.7003	28.7252	0.0250	28.7003	0.0000
3060	27.0539	27.0785	0.0246	27.0539	0.0000

We compared in table.2 and chart.2 analytically solution with Taylors method has error round about 0.0246 and compared with RK4 method gives occurs zero error by using Python

The RK4 method provides results almost identical to the analytical solution, with errors close to zero.

The Taylor method presents a small error that slowly increases over time but remains very small (~0.025°C at the largest time step).

Both methods closely approximate the analytical solution, making them reliable for practical purposes.

IV. CONCLUSION

This research effectively demonstrates the comparative performance of analytical and numerical methods in solving Newton's Law of Cooling through a time-dependent framework by PYTHON. By analyzing the temperature of an object cooling from 80°C in a refrigerated environment at 5°C, the study evaluates both the Taylor method and the Runge-Kutta 4th Order (RK4) method over a simulation period of 3000 or 3100 seconds. The results confirm that the analytical solution serves as a precise benchmark for evaluating numerical accuracy.

The Taylor method, though computationally simple, shows a gradual increase in error over time, making it less suitable for long-term simulations. In contrast, the RK4 method maintains exceptionally high accuracy and stability throughout, closely tracking the analytical solution at all time intervals. The use of time-dependent analysis emerges as a key strength of this study. It not only captures the dynamic behavior of the system but also highlights the rate and nature of error accumulation, which static comparisons fail to show.

This insight is essential for selecting appropriate numerical methods in practical scenarios, particularly when modeling temperature-dependent or time-evolving physical processes.In conclusion, this research confirms the value of RK4 for precision modeling in cooling problems and emphasizes that considering time-dependency is critical for understanding numerical behavior, ensuring accuracy, and guiding method selection in scientific computing.

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REFERENCES

1. Rehan, Z. (2020). Application of first-order differential equation to heat convection in fluid. Journal of Applied Mathematics and Physics, 8(8), 1456–146?.
2. Noar, N.A., Apandi, N.I., & Rosli, N. (2024). A comparative study of Taylor method, fourth-order Runge-Kutta method and Runge-Kutta Fehlberg method to solve ordinary differential equations. AIP Conference Proceedings, 2895(1).
3. Hassan, A., & Zakari, Y. (2018). Application of first-order differential equation in temperature problems. Applied Differential Equations, 16, 1–6.
4. Butcher, J.C. (1966). On the convergence of numerical solutions to ordinary differential equations. Mathematics of Computation, 20(93), 1–10.
5. Islam, M.A. (2015). A comparative study on numerical solutions of initial value problems (IVP) for ordinary differential equations (ODE) with Euler and Runge-Kutta methods. American Journal of Computational Mathematics, 5(3), 393–404.
6. Kamruzzaman, M., & Nath, M.C. (2018). A comparative study on numerical solution of initial value problem by using Euler's method, modified Euler's method and Runge-Kutta method. Journal of Computer and Mathematical Sciences, 9(5), 493–500.
7. Shawagfeh, N., & Kaya, D. (2004). Comparing numerical methods for the solutions of systems of ordinary differential equations. Applied Mathematics Letters, 17(3), 323–328.
8. Karthikeyan, N., & Jayaraja, A. (2015). Application of first-order differential equations to heat transfer analysis in solid. International Journal of Engineering and Innovative Technology, 5(8), 5–8.
9. Barrio, R., Blesa, F., & Lara, M. (2005). VSVO formulation of the Taylor method for the numerical solution of ODEs. Computers & Mathematics with Applications, 50(1–2), 93–111.
10. Corliss, G., & Chang, Y.F. (1982). Solving ordinary differential equations using Taylor series. ACM Transactions on Mathematical Software, 8(2), 114–144.
11. Caronongan, K. (n.d.). An application of differential equations in the study of elastic columns.
12. Corliss, G., & Lowery, D. (1977). Choosing a stepsize for Taylor series methods for solving ODEs. Journal of Computational and Applied Mathematics, 3(4), 251–256.
13. Ogunrinde, R.B., Fadugba, S.E., & Okunlola, J.T. (2012). On some numerical methods for solving initial value problems in ordinary differential equations. IOSR Journal of Mathematics, 25–31.
14. Amirul, I.M. (2015). Accurate solutions of initial value problems for ordinary differential equations with the fourth-order Runge-Kutta method. Journal of Mathematics Research, 7(3).
15. Karthikeyan, N., & Jayaraja, A. (2016). Application of first-order differential equation to heat transfer in solid. International Journal of Engineering and Innovative Technology.