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Kth Root Transformation for a Subclass of Analytic Functions Related to Cosine Function

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Abstract- The purpose of this paper is to obtain an upper bound for the third Hankel determinant corresponding to the root transformationfor a subclass of univalent analytic functions related to the Cosine function. 2010Mathematics Subject Classification:Primary 30C45, 30C50; Secondary 30C80.

Keywords- Analytic function, root transformation, Function with positive real part, Starlike function, Subordination,Cosine function, Hankel determinant.

I. INTRODUCTION

Let $U=\{z\in C:|z|<1\}$ be the open unit disk and $\mathcal{A}$ be the family of allanalyticfunctions f in U, with the normalized condition $f(0)=f'(0)-1=0$. The functions in $\mathcal{A}$ are of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n,\ \forall z\in U \qquad (1.1)$$

Let S be the subclass of $\mathcal{A}$ consisting ofunivalent functions of the form (1.1). The k^{th} root transformation of f is defined by

$$F(z)=[f(z^k)]^{\frac{1}{k}}=z+\sum_{n=1}^{\infty}b_{nk+1}z^{nk+1},\quad \forall k\in N \ (1.2)$$

Recently, Ali et al. [2] have introduced and studied the upper bounds of coefficient inequalities corresponding to the k^{th} root transformation.

A function $f\in S$ is said to be starlike if and only if,

$$Re\left\{\frac{zf'(z)}{f(z)}\right\}>0,\quad \forall z\in U \qquad (1.3)$$

The class of all functions p in U which are analytic with $p(0)=1$ and $\mathcal{Re}\{p(z)\}>0$ is denoted by P. A function $p\in P$ will be of the form

$$p(z)=1+\sum_{n=1}^{\infty}c_nz^n \qquad (1.4)$$

The class of all analytic functions $w(z)$ in U with $w(0)=0$ and $|w(z)|\leq 1$ is denoted by B_0 and the functions in this class are called as Schwarz functions. Let two functions f and g be analytic in U, if there exists a Schwarz function $w(z)$ in B_0 such that $f(z)=g(w(z))$, $\forall z\in U$ then we can say that f is subordinate to g and write it as $f\prec g$.

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Definition 1.1. Let $\phi(z)=\cos z$ be anunivalent analytic function, whichmaps from the unit disc onto a circular domain. It is symmetric with respect to real axis from origin to the point 1.41 on the real axis. It is a function with positive real part with $\phi(0)=1$ and $\phi'(0)=0$.

Definition 1.2. If f is in the class $S^*(\phi)$, then it satisfies the condition

$$\left(\frac{zf'(z)}{f(z)}\right)\prec \cos z \tag{1.5}$$

where the branch of the square root is chosen to be $\phi(0)=1$.

Noonan and Thomas [6] studied the q^{th} Hankel determinant of a sequence $a_n, a_{n+1}, a_{n+2}.....$of real or complex numbers as given below

$$H_q(n)=\begin{vmatrix} a_n & a_{n+1} & \cdots & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & \cdots & a_{n+2q-2} \end{vmatrix} \quad (n,q\epsilon N) \tag{1.6}$$

In particular, $H_2(1)$, $H_2(2)$, $H_3(1)$ are second and third Hankel determinants, respectively.

Babalola [3],Murugusundaramoorthy[5] andVamsheekrishna[13]have studied $H_3(1)$for different subclasses of analytic functions.The q^{th} Hankel determinant associated with the k^{th} root transformation of the function f is given by

$$\left[H_q(n)\right]^{\frac{1}{k}}=\begin{vmatrix} b_{nk} & b_{nk+1} & \cdots & \cdots & b_{(n+q-2)k+1} \\ b_{nk+1} & b_{(n+1)k+1} & \cdots & \cdots & b_{(n+q-1)k+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ b_{(n+q-2)k+1} & b_{(n+q-1)k+1} & \cdots & \cdots & b_{(n+2(q-1))k+1} \end{vmatrix} \quad (q,n,k\in\mathbb{N}) \tag{1.7}$$

For $n=1\&q=2$,the above equation (1.8)reduces to the coefficient functional of Fekete-Szegö related to the k^{th} root transformation which is given by

$$[H_2(1)]^{\frac{1}{k}}=\begin{vmatrix} b_k & b_{k+1} \\ b_{k+1} & b_{2k+1} \end{vmatrix}=b_{2k+1}-b_{k+1}^2 \quad (\because b_k=1)$$

For $n=2\&q=2$,the above equation (1.8) reduces to the coefficient functional called assecond Hankel determinantwhich is given by

$$[H_2(2)]^{\frac{1}{k}}=\begin{vmatrix} b_{2k} & b_{2k+1} \\ b_{2k+1} & b_{3k+1} \end{vmatrix}=b_{2k}b_{3k+1}-b_{2k+1}^2$$

For $n=1\&q=3$, the equation (1.8) reduces to the third Hankel determinant associated with the k^{th} root transformation of the function $f$given by

$$[H_3(1)]^{\frac{1}{k}}=\begin{vmatrix} b_k & b_{k+1} & b_{2k+1} \\ b_{k+1} & b_{2k+1} & b_{3k+1} \\ b_{2k+1} & b_{3k+1} & b_{4k+1} \end{vmatrix}$$

$$[H_3(1)]^{\frac{1}{k}}=b_{2k+1}\left[b_{k+1}b_{3k+1}-b_{2k+1}^2\right]-b_{3k+1}\left[b_kb_{3k+1}-b_{2k+1}b_{k+1}\right]+b_{4k+1}\left[b_kb_{2k+1}-b_{k+1}^2\right]$$

By taking $b_k=1$ and applying triangular inequality, we have

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$$\left|H_3(1)\right|^{\frac{1}{k}} \le \left|b_{2k+1}\right|\left|b_{k+1}b_{3k+1}-b_{2k+1}^2\right|+\left|b_{3k+1}\right|\left|b_{2k+1}b_{k+1}-b_{3k+1}\right|+\left|b_{4k+1}\right|\left|b_{2k+1}-b_{k+1}^2\right| \quad (1.9)$$

This determinant was studied by Sharma et al. [12] and Vamshee Krishna et al. [13,14,15].

Motivated by the works of Babalola [3],MohsanRaza [9],Sharma [12] andVamsheeKrishna [13,14,15],initial coefficient bounds, second and third-order Hankel determinants for a class of analytic univalent functions subordinated to $\cos z$ were studied by Yakaiah et al. [16], kth root transformations for certain subclasses of analytic functions were studied by Haripriya et al. [17, 18, 19].Wenow investigate the initial coefficient bounds, second and third-order Hankel determinants associated with k^{th} root transformation for the members of $S^*(\phi)$ where $\phi(z)=\cos z$.

II. PRELIMINARIES

To prove themain results in this paper, we need the following lemmas concerning the members of class P

Lemma 2.1.[7] If the function $p\in P$then $|c_n|\le 2\ \forall n\in N$.

Lemma 2.2.[4] If the function $p\in P$then

$$c_2=\frac{1}{2}\left\{c_1^2+x\left(4-c_1^2\right)\right\}$$

$$c_3=\frac{1}{4}\left\{c_1^3+2\left(4-c_1^2\right)c_1x-\left(4-c_1^2\right)c_1x^2+2\left(4-c_1^2\right)\left(1-|x|^2\right)z\right\}$$

For some x,z with $|x|\le 1$ and $|z|\le 1$

Lemma 2.3. [8] If $p(z)=1+c_1z+c_2z^2+c_3z^3+........=1+\sum_{n=1}^{\infty}c_nz^n$ where $z\in U$ then for any complex number υ we have $\left|c_2-\upsilon c_1^2\right|\le 2\max\{1,|2\upsilon-1|\}$

The sharpness is obtained by thefunctions $P(z)=\dfrac{1+z^2}{1-z^2}$ and $P(z)=\dfrac{1+z}{1-z}$.

Lemma 2.4. [8] If $p(z)=1+c_1z+c_2z^2+c_3z^3+........=1+\sum_{n=1}^{\infty}c_nz^n$ where $z\in U$ then

$\left|Jc_1^3-Kc_1c_2+Lc_3\right|\le 2\left(|J|+|K-2J|+|J-K+L|\right)$ for any real numbers J, K, L.

III. MAIN RESULTS

1. Initial Coefficient inequality for $f\in S^*(\phi)$

Theorem 3.1: If $f\in S^*(\phi)$and F is the k^{th} root transformation of f of the form (1.2) then

$$|b_{k+1}|=0, \quad (3.1)$$

$$|b_{2k+1}|\le\frac{1}{4k}, \quad (3.2)$$

$$|b_{3k+1}|\le\frac{1}{6k}, \quad (3.3)$$

$$|b_{4k+1}| \leq \frac{3}{8k}. \quad (3.4)$$

The first two inequalities are sharp.

Proof: If $f \in S^*(\phi)$, then by using Schwarz function and subordination principle, we have

$$\frac{zf'(z)}{f(z)} = \phi(w(z)) \quad (3.5)$$

Consider a function with positive real part defined as

$$p(z) = \frac{1+w(z)}{1-w(z)} = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \ldots\ldots\ldots \quad (3.6)$$

$$\Rightarrow w(z) = \frac{p(z)-1}{p(z)+1} \quad \text{for some } p \in P \quad (3.7)$$

By simple computations, we get

$$\frac{zf'(z)}{f(z)} = 1 + a_2 z + (2a_3 - a_2^2)z^2 + (3a_4 - 3a_2a_3 + a_2^3)z^3 + (4a_5 - 4a_2a_4 - 2a_3^2 - 4a_2^2a_3 - a_2^4)z^4 + \ldots \quad (3.7)$$

$$\cos(w(z)) = 1 - \left[\frac{c_1^2}{16}\right]z + \left[\frac{c_1^3}{8} - \frac{c_1c_2}{4}\right]z^3 + \ldots.. \quad (3.8)$$

From (3.5), (3.7) and (3.8),one can obtain

$$a_2 = 0, \quad (3.9)$$

$$a_3 = -\frac{c_1^3}{16}, \quad (3.10)$$

$$a_4 = \frac{c_1^3}{24} - \frac{c_1c_2}{12}, \quad (3.11)$$

$$a_5 = \frac{1}{4}\left[\frac{-c_1}{24}(2c_1^3 - 9c_1c_2 + 6c_3) - \frac{c_2^2}{8}\right]. \quad (3.12)$$

By applying k^{th} root transformation of the function f, we have

$$F(z) = [f(z^k)]^{\frac{1}{k}} = z + \sum_{n=1}^{\infty} b_{nk+1} z^{nk+1}$$

$$F(z) = z + \left[\frac{a_2}{k}\right]z^{k+1} + \left[\frac{a_3}{k} - \frac{(k-1)a_2^2}{2k^2}\right]z^{2k+1} + \left[\frac{a_4}{k} - \frac{(k-1)a_2a_3}{k^2} + \frac{(k-1)(2k-1)a_2^3}{6k^3}\right]z^{3k+1}$$
$$+ \left[\frac{a_5}{k} - \frac{(k-1)a_3^2}{2k^2} - \frac{(k-1)a_2a_4}{k^2} + \frac{(k-1)(2k-1)a_2^2a_3}{2k^3} - \frac{(k-1)(2k-1)(3k-1)a_2^4}{24k^4}\right]z^{4k+1} + \ldots\ldots$$

By comparing the coefficients of F(z) given above with (1.2) and using the relations (3.9) to (3.12), We have

$$b_{k+1} = 0, \quad (3.13)$$

$$b_{2k+1} = \frac{-c_1^2}{16k}, \quad (3.14)$$

$$b_{3k+1} = \frac{c_1}{24k}(c_1^2 - 2c_2), \quad (3.15)$$

$$b_{4k+1} = \frac{1}{1536k^2}\left(-c_1\left((35k-3)c_1^3 - 96k\,c_1c_2 + 96k\,c_3\right) - 48k\,c_2(c_2 - c_1^2)\right). \quad (3.16)$$

Taking modulus on both sides eqs. (3.13) – (3.16) followed by applying Lemmas (2.1) –(2.4), we obtain the required inequalities.

If we choose $p(z)=\frac{1+z}{1-z}$ in equation (3.6), then $F_1(z)=z+\frac{1}{4k}z^{2k+1}+\frac{k+3}{96k^2}z^{4k+1}+\ldots$ is an extremal function for the inequality for first two inequalities.

2. Second and Third Hankel determinantfor $f\in SL^*(\phi)$

Theorem 3.2: If $f\in S^*(\phi)$, then the second Hankel determinant corresponding to the k^{th} root transformation of f is given by

$$\left|b_{k+1}b_{3k+1}-b_{2k+1}^2\right|=\frac{1}{16k^2}\quad \forall k\in N \text{ and } \left|b_{2k+1}-b_{k+1}^2\right|=\frac{1}{4k}\ \forall k\in N.$$

Proof: Follows from the fact that $b_{k+1}=0$, $b_{2k+1}=\frac{-c_1^2}{16k}$ and $|c_1|\le 2$.

Further, the function $F_1(z)=z+\frac{1}{4k}z^{2k+1}+\frac{k+3}{96k^2}z^{4k+1}+\ldots$ is an extremal function these inequalities.

Theorem3.3: If $f\in S^*(\phi)$, then the third Hankel determinant corresponding to the k^{th} root transformation is given by $|H_3(1)|^{\frac{1}{k}}\le\frac{187k-9}{1152}\ \forall k\in N.$

Proof: Let $f\in S^*(\phi)$ and F be the k^{th} root transformation of the function f.

Then from the definition of $|H_3(1)|^{1/k}$ corresponding to k^{th} root transformation is given by with $b_k=1$ and $b_{k+1}=0$, we have

$$|H_3(1)|^{\frac{1}{k}}=\left|b_{4k+1}b_{2k+1}-b_{3k+1}^2-b_{2k+1}^3\right|$$

$$=\frac{1}{73728k^3}\left|c_1^3\left((-23k+9)c_1^3-288kc_1c_2+288kc_3\right)-368kc_1^2c_2\left(c_2-c_1^2\right)\right|$$

$$\le\frac{1}{73728k^3}\left(\left|(-23k+9)c_1^3-288kc_1c_2+288kc_3\right||c_1|^3+368k|c_1|^2|c_2||c_2-c_1^2|\right)$$

In view of Lemma 2.4, Lemma 2.3, and Lemma 2.1, we have

$$\left|(-23k+9)c_1^3-288kc_1c_2+288kc_3\right|\le 760k-72,\ \left|c_2-c_1^2\right|\le 2,\ |c_1|\le 2, |c_2|\le 2.$$

Hence, $|H_3(1)|^{\frac{1}{k}}\le\frac{1}{73728k^3}(6080k-576+5888k)=\frac{11968k-576}{73728k^3}=\frac{187k-9}{1152}$.

3.3. Coefficient functionalrelated with $\frac{z}{f(z)}$

The k^{th} root transformation for the function $\frac{z}{f(z)}$ is given by $G(z)=\left[\frac{z^k}{f(z^k)}\right]^{\frac{1}{k}}=1+\sum_{n=1}^{\infty}d_{nk}z^{nk}$

Theorem 3.4. If $f\in SL^*(\phi)$ and $G(z)=\left[\frac{z^k}{f(z^k)}\right]^{\frac{1}{k}}$, then for any complex number μ, we have

$$\left|d_{2k}-\mu d_k^2\right|\le\frac{1}{4k}\max\left\{1,\left|\frac{k+2-4\mu}{4k}\right|\right\}.$$

Proof: Let $G(z)=\left[\frac{z^k}{f(z^k)}\right]^{\frac{1}{k}}=\left[1+\sum_{n=1}^{\infty}b_{nk+1}z^{nk}\right]^{-1}$

$$1+\sum_{n=1}^{\infty}d_{nk}z^{nk}=1-b_{k+1}z^k+\left[b_{k+1}^2-b_{2k+1}\right]z^{2k}+\left[2b_{k+1}b_{2k+1}-b_{3k+1}-b_{k+1}^3\right]z^{3k}$$
$$+\left[2b_{k+1}b_{3k+1}+b_{2k+1}^2-b_{4k+1}-3b_{k+1}^2b_{2k+1}+b_{k+1}^4\right]z^{4k}+\cdots\cdots$$

Comparing the coefficients of similar powers of z and using the values of b_{k+1} and b_{2k+1} from (3.13) - (3.16), we obtain

$$d_k=-b_{k+1}=0\ d_{2k}=b_{k+1}^2-b_{2k+1}=\frac{c_1^2}{16k}$$

$$\Rightarrow d_{2k}-\mu d_k^2=\frac{c_1^2}{16k}$$

Taking modulus on both sides and using Lemma 2.3, we obtain $\left|d_{2k}-\mu d_k^2\right|=\left|\frac{c_1^2}{16k}\right|\leq\frac{1}{4k}.$

IV. CONCLUSION

In this paper, we introduced and investigated a subclass of analytic functions associated with the cosine function through the application of a root transformation operator. Various geometric properties of this class were established by employing the theory of subordination and techniques from geometric function theory. We derived coefficient estimates and examined the influence of the root transformation on the analytic and geometric behavior of the functions under consideration.

The obtained results demonstrate that the root transformation preserves several important characteristics of the underlying cosine-related function class while generating new families of analytic functions with interesting geometric properties. The findings presented here generalize and extend several existing results available in the literature for related subclasses of univalent functions.

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