# Estimation of Reliability in a Consecutive linear/circular *k*-out-of-*n* system based on Weighted Exponential-Lindley distribution

**[1]Sunita Sharma; [2]Vinod Kumar**

1Department of Mathematics, Statistics and Computer Science,

G.B. Pant University of Agriculture and Technology, Pantnagar, India

03sharmasunita@gmail.com

2Department of Mathematics, Statistics and Computer Science,

G.B. Pant University of Agriculture and Technology, Pantnagar, India

vinod_kumarbcb@yahoo.com

**Abstract**

Consecutive *k*-out-of-*n* systems have gained significant attention and found diverse applications across various domains. This research article introduces a Classical and Bayesian approach for reliability estimation in Consecutive linear/circular *k*-out-of-*n*: F systems using the Weighted Exponential-Lindley distribution. By employing this distribution to model component lifetimes, we obtained maximum likelihood and Bayesian estimates for reliability using squared error loss function. In cases where exact forms are unattainable, Lindley's approximation and the Markov chain Monte Carlo method are utilized to derive Bayes estimates. We also examined mean time to failure and constructed credible intervals to estimate Bayes reliability. To assess and compare the effectiveness of these estimators, we carried out a Monte Carlo simulation study.

**Keywords:** Consecutive *k*-out-of-*n* system; Bayesian estimates; reliability, squared error loss function; Weighted Exponential-Lindley distribution; credible intervals

## I. Introduction

Consecutive *k*-out-of-*n* systems have experienced remarkable progress and have found extensive applications across various domains, owing to their exceptional reliability and fault tolerance capabilities. The concept of the consecutive-*k* system was initially introduced by Kontoleon [1] and subsequently, Chiang and Niu [2] coined the term "consecutive-*k*-out-of-*n*: F (cons/*k*-*n*: F) system" to refer to a specific type within this framework. In a cons/*k*-*n*: F system, failure occurs only when *k* or more components fail consecutively out of a total of *n* components. Consecutive-*k*-out-of-*n* systems are frequently encountered in various engineering applications, where assessing the system reliability becomes a crucial concern. These systems find relevance in diverse domains such as telecommunications, microwave relay stations, oil pipeline systems, vacuum systems in accelerators, computer ring networks, and spacecraft relay stations. Depending on the logical or physical connections among components, these systems can be categorized as either linear or circular, while their functioning principle is classified as either failed (F) or good (G). In the consecutive-*k*-out-of-*n*: F system, comprising a linear or circular arrangement of *n* components, failure occurs when *k* consecutive components experience failure. When the components of the cons/*k*-*n*: F system are positioned in a linear configuration, it is referred to as a linear consecutive *k*-out-of-*n*: F (L(cons/*k*-*n*:

F)) system. Conversely, if the components are arranged in a circular fashion, it is known as a circular consecutive $k$-out-of-$n$: F (C(cons/$k$-$n$: F)) system. In the L(cons/$k$-$n$: F) system, the first and last components are not adjacent to each other in terms of consecutiveness. However, in the C(cons/$k$-$n$: F) system, the first and last components form a consecutive pair. Extensive research has been conducted on consecutive-$k$-out-of-$n$ systems in the existing literature, building upon the initial work by Kontoleon [1]. Notable references for further exploration include studies by Fu [3], Bollinger and Salvia [4], Zuo and Kuo [5], Chang *et al.* [6], Kuo and Zuo [7], Eryilmaz [8], Eryilmaz [9], Gokdere and Gurcan [10], Guan and Wu [11], Wang *et al.* [12], Yuan and Cui [13] and Hongda *et al.* [14]. Derman *et al.* [15] introduced a formula for calculating the reliability of an L(cons/k-n: F) system composed of i.i.d. components with reliability $p$ given by

$$R(k,n,p) = \sum_{j=0}^{n} N(j, n-j+1, k-1) q^j \, p^{n-j} \qquad (1)$$

Lambiris and Papastavridis [16] derived an expression for the numbers $N(j, n-j+1, k-1)$ in equation (2), with the condition $n-j+1 \geq 0$ as follows

$$N(j, n-j+1, k-1) = \sum_{\lambda=0}^{n-j+1} \binom{n-j+1}{\lambda}\binom{n-\lambda k}{j-\lambda k}(-1)^{\lambda} \qquad (2)$$

Consecutive $k$-out-of-$n$ systems pose a significant challenge in reliability estimation due to their specific failure criterion. The requirement of consecutive failures introduces a new layer of complexity, as traditional methods may struggle to accurately assess the reliability of such systems. To address these challenges, we propose a Bayesian framework for reliability estimation in consecutive $k$-out-of-$n$ systems. Bayesian methods offer several advantages over frequentist approaches, as they allow for the incorporation of prior knowledge, explicit modeling of uncertainties, and updating of beliefs based on observed data. By leveraging Bayesian techniques, we can obtain more accurate reliability estimates while quantifying the associated uncertainties. Unlike frequentist approaches that often provide point estimates without considering uncertainty, Bayesian methods enable the calculation of credible intervals or posterior distributions that provide a more comprehensive understanding of the reliability estimates. This additional information aids decision-making processes by considering the level of confidence or uncertainty associated with the reliability estimates. The construction of a Bayesian confidence interval involves determining the lower and upper bounds that enclose the specified credibility level. This can be done by computing the quantiles of the posterior distribution. For example, a 95% Bayesian confidence interval would correspond to the 2.5th and 97.5th percentiles of the posterior distribution.

In recent years, there has been a notable surge of research interest in employing Bayesian methodology for the estimation of reliability in consecutive $k$-out-of-$n$ systems. Notably, Madhumitha and Vijayalakshmi [17] undertook a comprehensive study focusing on Bayesian reliability estimates of consecutive $k$-out-of-$n$: F system, utilizing the Weibull distribution. Eryilmaz and Navarro [18] discussed the failure rates of consecutive $k$-out-of-$n$ systems in their study. Yin and Cui [19] investigated the concept of reliability in a consecutive- $k$ -out-of- $n$: F system that incorporates shared components between adjacent subsystems. Further, Madhumitha and Vijayalakshmi [20] focused on estimating the Bayesian system reliability of a consecutive $k$-out-of-$n$: F system. They employed the negative binomial distribution for their analysis. Madhumitha and Vijayalakshmi [21] conducted a study on Bayesian Estimation of Linear/Circular Consecutive $k$-out-of-$n$: F System Reliability. Demiray and Kızılaslan [22] provided estimates of the reliability of a consecutive linear $k$-out-of-$n$ system comprising non-identical strength components. They specifically applied their methodology to wind speed data analysis. Also, Demiray and Kızılaslan [23] provided estimate of stress strength reliability of a consecutive $k$-out-of-$n$ system based on proportional hazard rate family respectively.

This paper investigates the Classical and Bayesian approach for estimating reliability in a consecutive linear/circular $k$-out-of-$n$: F system. To achieve this, we propose the utilization of the Weighted Exponential-Lindley distribution (WXLD) which allows us to incorporate both partial data and previous experience into our reliability assessment. The adoption of Bayesian inference is motivated by its remarkable ability to handle uncertainties inherent in reliability estimation. To model the lifetime of components within the system accurately, we leverage the flexibility provided by the WXLD. This distribution enables us to capture various patterns of component failures, ensuring a comprehensive analysis of the system's reliability. By utilizing this distribution, we derive a maximum likelihood estimate and Bayesian estimate of the proposed system using SELF. Moreover, we present Bayesian estimators for additional performance measures, such as mean time to failure (MTTF) and Credible intervals. To illustrate the practical applicability of our developed model, we provide a numerical example that effectively demonstrates its effectiveness in estimating reliability and related performance measures. This paper is organized as follows: Model description is given in section 2. In section 3, background details including notations, assumption and system reliability are explained. Classical and Bayesian reliability estimation is provided in section 4, followed by simulation study in section 5. Finally, in section 6 the results are concluded.

## II. Model Description

Chouia and Zeghdoudi [24] introduced Exponential-Lindley distribution (XLD), which is a unique distribution formed by combining two separate distributions the Exponential and the Lindley distribution. In this study, we proposed a distribution, called Weighted Exponential-Lindley distribution (Sharma and Kumar [25]), which is a mixture of gamma (2, 1/θ) and one-parameter XLD and it is described as follows: Let a random variable $X \sim WXLD(\theta)$ then the probability density function PDF and Cumulative distribution function (CDF) are

$$f(x) = \frac{4\theta^3 x(2+\theta+x)e^{-2\theta x}}{(1+\theta)^2} \; ; \; x \geq 0, \theta > 0 \qquad (3)$$

$$\Phi(x) = 1 - e^{-2\theta x}\left(\frac{2\theta^2 x^2}{(1+\theta)^2} + 2\theta x + 1\right) \qquad (4)$$

# III. Background

### 3.1 Notations

| | |
|---|---|
| $\theta$ | Failure rate |
| $\hat{\theta}$ | MLE of $\theta$ |
| i.i.d. | independent an identically distributed |
| cons/$k$-$n$: F | consecutive $k$-out-of-$n$: F system |
| L(cons/$k$-$n$: F) | Linear cons/$k$-$n$: F |
| C(cons/$k$-$n$: F) | Circular cons/$k$-$n$: F |
| $R_c(t)$ | Component reliability |
| $R_s(t)$ | System reliability |
| $\hat{R}_c(t)$ | MLE of component reliability |
| $\hat{R}_s(t)$ | MLE of system reliability |
| $\hat{R}_s^L(t)$ | MLE of Linear cons/$k$-$n$: F |
| $\hat{R}_s^C(t)$ | MLE of Circular cons/$k$-$n$: F |
| $R_s^L(t)$ | Reliability of L(cons/$k$-$n$: F) |
| $R_s^{L*}(t)$ | Bayes estimate for reliability of L(cons/$k$-$n$: F) |
| $R_s^C(t)$ | Reliability of C(cons/$k$-$n$: F) |
| $R_s^{C*}(t)$ | Bayes estimate for reliability of C(cons/$k$-$n$: F) |
| $\mu_L$ | MTTF of L(cons/$k$-$n$: F) |
| $\mu_L^*$ | Bayes estimate of MTTF for L(cons/$k$-$n$: F) |
| $\mu_C$ | MTTF of C(cons/$k$-$n$: F) |
| $\mu_C^*$ | Bayes estimate of MTTF for C(cons/$k$-$n$: F) |

### 3.2 Assumptions

**1.** At time $t = 0$, all components are in a good state and functioning properly.

**2.** There are $n$ identical components that are functioning properly.

**3.** The component can be either operational or in a state of failure.

**4.** Perfect links and connections are assumed in the system.

**5.** The components exhibit mutual independence and identical distribution.

**6.** The failure time of the component follows WXLD.

**7.** The system experiences a failure when at least $k$ consecutive components fail, where $k$ is less than or equal to $n$.

### 3.3. System Reliability

Consider a L(cons/k-n: F) system consists of *n* components that fail whenever *k* components fail consecutively, with *k* being less than or equal to *n*. In contrast, the C(cons/k-n: F) system arranges the components in a circle, where the first and last components are consecutive. The reliability of a L(cons/*k*-*n*: F) and C(cons/*k*-*n*: F) system is presented by Kuo and Zuo [7]. We obtained the following reliability function and the result derived by Griffith and Govindarajulu [26].

The reliability function of L(cons/*k*-*n*: F) system is given by

$$R_s^L(t) = \sum_{l=0}^{N1} (-1)^l \, C_{N3}^l R_c^l (1-R_c)^{N4} - \sum_{l=0}^{N1} (-1)^l C_{N5}^l \, R_c^l (1-R_c)^{N6} \tag{5}$$

where, $N1 = \left\lfloor \frac{n}{k+1} \right\rfloor$, $N3 = n - lk$, $N4 = kl$, $N5 = n - lk - k$, $N6 = kl + k$

The reliability function of C(cons/*k*-*n*: F) system is given by

$$R_s^C(t) = \sum_{l=0}^{N1} (-1)^l \, C_{N3}^l R_c^l (1-R_c)^{N4} - \sum_{l=0}^{N2} (-1)^{l+1} C_{N7}^l \, R_c^{l+1} (1-R_c)^{N6} - (1-R_c)^n \tag{6}$$

where, $N1 = \left\lfloor \frac{n}{k+1} \right\rfloor$, $N2 = \left\lfloor \frac{n}{k+1} - 1 \right\rfloor$, $N3 = n - lk$, $N4 = kl$, $N5 = n - lk - k$, $N6 = kl + k$, $N7 = n - lk - k - 1$

Further, we assume that the lifetime of the component be WXLD with PDF given in equation (3).

and for a mission time $t$ the reliability of each component of the considered system is

$$R_c(t) = e^{-2\theta t} \left( \frac{2\theta^2 t^2}{(1+\theta)^2} + 2\theta t + 1 \right), t \geq 0, \theta \geq 0 \tag{7}$$

## IV. Estimation of System Reliability

In this section, we focus on estimating both $R_s^L$ and $R_s^C$. We derive the Maximum likelihood estimates (MLE) and Bayes estimates for $R_s^L$ and $R_s^C$.

### 4.1 MLE of $R_s^L$ and $R_s^C$

Suppose that *n* units are initially placed on test and terminating the test once all *n* units are failed. The failure times of these units, denoted by $t_1, t_2, \ldots\ldots, t_n$ are assumed to be distributed with density function as given in equation (3) that depends on a single parameter $\theta$

The Likelihood function of the data $t_1, t_2, \ldots\ldots, t_n$ is expressed as,

(8)

$$L = L(\theta/t_1, t_2, \ldots\ldots, t_n) = \prod_{i=1}^{n} \frac{4\theta^3 t_i(2+\theta+t_i)e^{-2\theta t_i}}{(1+\theta)^2}$$

The log-likelihood function may be written as

$$l = nlog4\theta^3 - 2nlog(1+\theta) + \sum_{i=1}^{n} logt_i + \sum_{i=1}^{n} \log(2+\theta+t_i) - 2\theta \sum_{i=1}^{n} t_i \tag{9}$$

For calculating the MLE of parameter $\theta$, we partially differentiate equation (9) with respect to $\theta$ and equating to zero as

$$\frac{\partial l}{\partial \theta} = \frac{3n}{\theta} - \frac{2n}{(1+\theta)} + \sum_{i=1}^{n} \frac{1}{(2+\theta+t_i)} - 2\sum_{i=1}^{n} t_i = 0 \tag{10}$$

The MLE, $\hat{\theta}$ of parameter $\theta$ is the solution of equation (10). Due to the unavailability of a closed-form solution, a numerical iteration method has been employed to estimate the value of the parameter.

Now, by using the well-known Invariance property of MLE, the ML estimates of reliability function of components and cons/*k*-*n*: F system are respectively obtained as,

$$\hat{R}_c(t) = e^{-2\hat{\theta}t}\left(\frac{2\hat{\theta}^2 t^2}{(1+\hat{\theta})^2} + 2\hat{\theta}t + 1\right) \tag{11}$$

$$\hat{R}_s^L(t) = \sum_{l=0}^{N1} (-1)^l C_{N3}^l \hat{R}_c^l (1-\hat{R}_c)^{N4} - \sum_{l=0}^{N1} (-1)^l C_{N5}^l \hat{R}_c^l (1-\hat{R}_c)^{N6} \tag{12}$$

$$\hat{R}_s^C(t) = \sum_{l=0}^{N1} (-1)^l C_{N3}^l \hat{R}_c^l (1-\hat{R}_c)^{N4} - \sum_{l=0}^{N2} (-1)^{l+1} C_{N7}^l \hat{R}_c^{l+1} \left(1-\hat{R}_c\right)^{N6} - (1-\hat{R}_c)^n \tag{13}$$

### 4.2 Bayes Estimate of $R_s^L$ and $R_s^C$

In this part of the proposed study, we discuss the Bayesian reliability estimate of the considered L (cons/*k*-*n*: F)/ C (cons/*k*-*n*: F) system under SELF. We assume that the parameter $\theta$ follows gamma $(\alpha, \beta)$ prior distribution with PDF

$$g(\theta) = \frac{\beta^\alpha e^{-\theta\beta} \theta^{\alpha-1}}{\Gamma\alpha} \tag{14}$$

The likelihood function for the probability of $t$ given $\theta$, where the component failure rate $\theta$ follows a WXLD, can be expressed as

$$L(t/\theta) = \prod_{i=1}^{n} f(t_i/\theta) = \frac{\left(4\theta^3\right)^n \prod_{i=1}^{n} t_i(2+\theta+t_i) e^{-2\theta \sum_{i=1}^{n} t_i}}{(1+\theta)^{2n}} \tag{15}$$

Thus, the posterior distribution of $\theta$ under considered prior is found to be

$$\pi(\theta/t) = \frac{L(t/\theta)g(\theta)}{\int_0^{\infty} L(t/\theta)g(\theta)d\theta}$$

$$\pi(\theta/t) = \frac{\frac{(4\theta^3)^n \prod_{i=1}^{n} t_i(2+\theta+t_i)e^{-2\theta\sum_{i=1}^{n} t_i}\frac{\beta^\alpha e^{-\theta\beta}\theta^{\alpha-1}}{\Gamma\alpha}}{(1+\theta)^{2n}}}{\int_0^{\infty} \frac{(4\theta^3)^n \prod_{i=1}^{n} t_i(2+\theta+t_i)e^{-2\theta\sum_{i=1}^{n} t_i}}{(1+\theta)^{2n}}\frac{\beta^\alpha e^{-\theta\beta}\theta^{\alpha-1}}{\Gamma\alpha}d\theta} \tag{16}$$

Under SELF, the Bayes estimator of the reliability function $R_s^L(t)$ is given by:

$$R_s^{L*}(t) = E\left(R_s^L(t)\right) = \int_0^{\infty} R_s^L(t)\Pi(\theta/u)\, d\theta$$

$$=\int_0^{\infty} \sum_{l=0}^{N1}(-1)^l\, C_{N3}^l R_c^l(1-R_c)^{N4} - \sum_{l=0}^{N1}(-1)^l C_{N5}^l\, R_c^l(1-R_c)^{N6}\Pi(\theta/u)\, d\theta \tag{17}$$

where, $\pi(\theta/t)$ is derived in equation (16).

Further, the Bayes estimator of $R_s^C(t)$ is expressed as

$$R_s^{C*}(t) = E\left(R_s^C(t)\right) = \int_0^{\infty} R_s^C(t)\Pi(\theta/u)\, d\theta$$

$$= \sum_{l=0}^{N1}(-1)^l\, C_{N3}^l R_c^l(1-R_c)^{N4} - \sum_{l=0}^{N2}(-1)^{l+1} C_{N7}^l\, R_c^{l+1}(1-R_c)^{N6} - (1-R_c)^n\Pi(\theta/u)\, d\theta \tag{18}$$

To approximate the integral in equation (17) and (18), two alternative approaches are employed due to its lack of analytic computation. Lindley's approximation and the Markov Chain Monte Carlo (MCMC) method are utilized as alternative methods for approximating the integral.

**4.2.1 Lindley's Approximation**

Lindley [27] proposed an approximation method for computing the ratio of two integrals. This technique can be applied to calculate the posterior expectation of any arbitrary function, also it can simplify the computation of complex integrals by expressing them as a ratio of simpler integrals. Let $u(\theta)$ be any arbitrary function, then it's posterior expectation is expressed as,

$$E(u(\theta)/t) = \frac{\int u(\theta)v(\theta)e^{l(\theta)}d\theta}{\int v(\theta)e^{l(\theta)}d\theta} \tag{19}$$

where, $u(\theta)$ is the function of $\theta$ only, $v(\theta)$: prior density function and $l(\theta) =$ log likelihood function.

Using the Lindley's approximation, $E(u(\theta)/t)$ approximately estimated by

$$E(u(\theta)/t) = [u + \frac{1}{2}\sum_i\sum_j\left(u_{ij} + 2u_i\rho_j\right)\sigma_{ij} + \frac{1}{2}\sum_i\sum_j\sum_k\sum_l L_{ijk}\sigma_{ij}\sigma_{kl}u_l] + o\left(\frac{1}{n^2}\right) \qquad (20)$$

Here $i,j,k,l = 1,2,\ldots,m$ ; $\theta = (\theta_1,\theta_2,\ldots,\theta_m)$; $u_i = \frac{\partial u}{\partial\theta_i}$ ; $u_{ij} = \frac{\partial^2 u}{\partial\theta_i\partial\theta_j}$; $L_{ij} = \frac{\partial^2 l}{\partial\theta_i\theta_j}, \rho_i = \frac{\partial\rho}{\partial\theta_i}$ where $\rho$ is the logarithm of prior distribution.

By considering the one-parameter WXLD, the following equation can be derived:

$$E(u(\theta)/t) = u + \tfrac{1}{2}(u_{11}\sigma_{11}) + u_1\rho_1\sigma_{11} + \tfrac{1}{2}(L_{111}u_1\sigma_{11}^2) \qquad (21)$$

In this case,

$\rho_1 = \frac{(\alpha-1)}{\theta} - \beta$

$L_{11} = \frac{\partial^2 l}{\partial\theta^2} = \frac{-3n}{\theta^2} + \frac{2n}{(1+\theta)^2} - \sum_{i=1}^n \frac{1}{(2+\theta+t_i)^2}$

$L_{111} = \frac{6n}{\theta^3} - \frac{4n}{(1+\theta)^3} + 2\sum_{i=1}^n \frac{1}{(2+\theta+t_i)^3}$

$\sigma_{ij}, i,j = 1,2$ are obtained by using $L_{ij}, i,j = 1,2$.

$\sigma_{11} = [-L_{11}]^{-1} = \left[\frac{-3n}{\theta^2} + \frac{2n}{(1+\theta)^2} - \sum_{i=1}^n \frac{1}{(2+\theta+t_i)^2}\right]^{-1}$

Now, to obtain the Bayes estimator of $R_S^L$ and $R_S^C$ using Lindley's approximation, denoted as $[R_S^{L*}(t)]_{Lin}$ and $[R_S^{C*}(t)]_{Lin}$ the following procedure is followed:

$$[R_S^{L*}(t)]_{Lin} = R_S^L + \frac{1}{2}(R_{11}^L\sigma_{11}) + R_1^L\rho_1\sigma_{11} + \frac{1}{2}(L_{111}R_1^L\sigma_{11}^2) \qquad (22)$$

and, $$[R_S^{C*}(t)]_{Lin} = R_S^C + \tfrac{1}{2}(R_{11}^C\sigma_{11}) + R_1^C\rho_1\sigma_{11} + \tfrac{1}{2}(L_{111}R_1^C\sigma_{11}^2) \qquad (23)$$

Here, all the parameters are evaluated at $\hat{\theta}$. Also, $R_1^L = \frac{\partial R_S^L}{\partial\theta_1}$, $R_{11}^L = \frac{\partial^2 R_S^L}{\partial\theta_1^2}$ and $R_1^C = \frac{\partial R_S^C}{\partial\theta_1}$, $R_{11}^C = \frac{\partial^2 R_S^C}{\partial\theta_1^2}$.

**4.2.2 MCMC Method**

The implementation of MCMC techniques typically requires the use of the Metropolis-Hastings sampler. The Metropolis-Hasting (MH) algorithm, originally introduced by Metropolis *et al.* [28], can be employed as a solution. The algorithm for incorporating Metropolis-Hastings (MH) within Gibbs sampling is as follows:

1. Start with initial guess $\theta^{(0)}$.

2. Set $i = 1$.

3. To generate $\theta^{(i)}$ from $\pi(\theta^{(i-1)}/t)$ using the following MH algorithm, employ a normal proposal distribution $N(\theta^{(i-1)}, var(\theta))$.

4. Obtain a proposal $\theta^*$ from $N(\theta^{(i-1)}, var(\theta))$.

(i) Compute the acceptance probabilities $\tau_\theta = min\left[1, \frac{\pi(\theta^*/t)}{\pi(\theta^{(i-1)}/t)}\right]$.

(ii) Generate a $u_1$ from a Uniform (0,1) distribution.

(iii) If $u_1 < \tau_\theta$, accept the proposal and set $\theta^{(i)} = \theta^*$, else set $\theta^{(i)} = \theta^{(i-1)}$.

5. Evaluate the $(R_s^L)^{(i)}$ and $(R_s^C)^{(i)}$ at $\theta^{(i)}$.

6. Set $i = i + 1$.

7. To obtain the posterior sample $(R_s^L)^{(i)}$and $(R_s^C)^{(i)}$,repeat Steps 3 to 5, *N* times; $i = 1, 2, \dots N$.

The given sample is utilized for computing the Bayes estimate and constructing the credible interval (CRI) for $R_s^L$ and $R_s^C$ respectively. To ensure convergence and mitigate the impact of initial value selection, the first *M* simulated varieties are discarded. Subsequently, the selected samples $\theta^{(i)}$ where $i = M + 1 \dots N$, with N being sufficiently large, are utilized. Then, the Bayes estimate of $R_s^L$ and $R_s^C$ under a SELF is respectively given by

$$[R_s^{L*}(t)]_{MC} = \frac{1}{N-M} \sum_{i=M+1}^{N} (R_s^L)^{(i)} \tag{24}$$

$$[R_s^{C*}(t)]_{MC} = \frac{1}{N-M} \sum_{i=M+1}^{N} (R_s^C)^{(i)} \tag{25}$$

Then, the $100(1-\delta)\%$ CRIs for $R_s^{L*}$ and $R_s^{C*}$ respectively are determined by the method of Chen and Shao [29].

**4.3 Bayesian Estimate of MTTF**

The Bayes estimate of MTTF for a considered L(cons/*k*-*n*: F) system is expressed as

$$\mu_L^* = \int_0^\infty R_s^{L*}(t)dt$$

$$\mu_L^* = \int_0^\infty \int_0^\infty \sum_{l=0}^{N1} (-1)^l C_{N3}^l R_c^l (1-R_c)^{N4} - \sum_{l=0}^{N1} (-1)^l C_{N5}^l R_c^l (1-R_c)^{N6} \pi(\theta/t)\, d\theta dt \tag{26}$$

Similarly, the Bayes estimate of MTTF for a considered C(cons/*k*-*n*: F) system is given by

$$\mu_C^* = \int_0^\infty \int_0^\infty \sum_{l=0}^{N1} (-1)^l C_{N3}^l R_c^l (1-R_c)^{N4} - \sum_{l=0}^{N2} (-1)^{l+1} C_{N7}^l R_c^{l+1} (1-R_c)^{N6} - (1-R_c)^n \pi(\theta/t)\, d\theta dt \tag{27}$$

where, $\pi(\theta/t)$ is provided in equation (16).

The analytical solution of the equation (26) and (27) is not feasible. Hence, we use numerical methods which provide an effective approach for approximating the integrals.

## V. Simulation Study

In this part, Monte Carlo simulations are employed to compare the system reliability estimates between MLE and Bayesian estimation approaches. The estimates are accompanied by their respective mean square error (MSE) or estimated risks (ERs) values as well as biases. The performances of the point estimates are assessed by using MSE for MLE and ER for Bayesian estimates. The ER of $\theta$, when $\theta$ is estimated by $\hat{\theta}$ is given by

$$(ER)_\theta = \frac{1}{N}\sum_{i=1}^{N}\left(\hat{\theta}_i - \theta_i\right)^2 \tag{28}$$

under SELF. Also,

$$Bias\ (R_S^L) = \frac{1}{N}\sum_{i=1}^{N}[\left(\hat{R}_S^L\right)^{(i)} - R_S^L\ ]^2 \tag{29}$$

The computations for all analyses were conducted using MATLAB and R software, with a total of 2500 replications. The results presented are derived from these computational runs. We calculate the MSE based-on replication (*N*) as follows:

$$MSE(R_S^L) = \frac{1}{N}\sum_{i=1}^{N}\left[\left(\hat{R}_S^L\right)^{(i)} - R_S^L\ \right]^2 \tag{30}$$

For 2500 replications, the MLE of $R_S^L$ and $R_S^C$ are computed for sample sizes of *m* =10, 15, 20, and 25. The parameter $\theta$ was assigned values of 0.5, 1.5, and 2.0. Table 1-6 provides ML and Bayes estimates of $R_S^L$ and $R_S^C$ for various sample sizes (*m* = 10, 15, 20, and 25) across different values of the parameter $\theta$. The accompanying Table 7 presents the corresponding credible intervals constructed for these estimates.

**Table 1.** Estimates of $R_S^L$ when $\theta = 0.5$ for Gamma prior in L(cons/*k*-*n*: F) system

| | | | MLE | | | Bayes estimate | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| (*k*, *n*) | $R_S^L$ | *m* | $\hat{R}_S^L$ | Bias | MSE | $[R_S^{L*}]_{Lin}$ | Bias | ER | $[R_S^{L*}]_{MC}$ | Bias | ER |
| (3, 10) | 0.7224 | 10 | 0.7286 | 0.0062 | 0.0085 | 0.6974 | -0.0250 | 0.0032 | 0.7102 | -0.0122 | 0.00375 |
| | | 15 | 0.7277 | 0.0053 | 0.0061 | 0.7021 | -0.0203 | 0.0029 | 0.7121 | -0.0103 | 0.00321 |
| | | 20 | 0.7257 | 0.0033 | 0.0048 | 0.7123 | -0.0101 | 0.0026 | 0.7145 | -0.0079 | 0.00287 |
| | | 25 | 0.7268 | 0.0044 | 0.0035 | 0.7162 | -0.0062 | 0.0023 | 0.7178 | -0.0046 | 0.00245 |
| (4, 10) | 0.5662 | 10 | 0.5882 | 0.0220 | 0.0106 | 0.5451 | -0.0211 | 0.0041 | 0.5732 | 0.0070 | 0.00452 |
| | | 15 | 0.5827 | 0.0165 | 0.0075 | 0.5578 | -0.0084 | 0.0031 | 0.5780 | 0.0118 | 0.00354 |
| | | 20 | 0.5769 | 0.0107 | 0.0051 | 0.5604 | -0.0058 | 0.0028 | 0.5782 | 0.0120 | 0.00287 |
| | | 25 | 0.5754 | 0.0092 | 0.0042 | 0.5658 | -0.0004 | 0.0021 | 0.5521 | -0.0141 | 0.00371 |
| (7, 10) | 0.3123 | 10 | 0.3254 | 0.0131 | 0.0063 | 0.2565 | -0.0558 | 0.0045 | 0.2911 | -0.0212 | 0.00201 |
| | | 15 | 0.3213 | 0.0090 | 0.0034 | 0.2851 | -0.0272 | 0.0012 | 0.2927 | -0.0196 | 0.00187 |
| | | 20 | 0.3178 | 0.0055 | 0.0025 | 0.3012 | -0.0111 | 0.0010 | 0.2915 | -0.0208 | 0.00145 |
| | | 25 | 0.3165 | 0.0042 | 0.0021 | 0.3005 | -0.0118 | 0.0052 | 0.2912 | -0.0211 | 0.00132 |
| (9, 10) | 0.2062 | 10 | 0.2231 | 0.0169 | 0.0035 | 0.1789 | -0.0273 | 0.0058 | 0.2132 | 0.0070 | 0.00123 |
| | | 15 | 0.2142 | 0.0080 | 0.0026 | 0.1825 | -0.0237 | 0.0009 | 0.2013 | -0.0049 | 0.00097 |
| | | 20 | 0.2117 | 0.0055 | 0.0015 | 0.1845 | -0.0217 | 0.0005 | 0.2086 | 0.0024 | 0.00084 |
| | | 25 | 0.2132 | 0.0070 | 0.0012 | 0.1912 | -0.0150 | 0.0004 | 0.2116 | 0.0054 | 0.00073 |

**Table 2.** Estimates of $R_s^L$ when $\theta = 1.5$ for Gamma prior in L(cons/*k*-*n*: F) system

| | | | MLE | | | Bayes estimate | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $(k, n)$ | $R_s^L$ | $m$ | $\hat{R}_s^L$ | Bias | MSE | $[R_s^{L*}]_{Lin}$ | Bias | ER | $[R_s^{L*}]_{MC}$ | Bias | ER |
| (3, 10) | 0.5141 | 10 | 0.5291 | 0.0050 | 0.0125 | 0.5265 | 0.0124 | 0.0084 | 0.5271 | 0.0130 | 0.00861 |
| | | 15 | 0.5287 | 0.0046 | 0.0074 | 0.5245 | 0.0104 | 0.0060 | 0.5248 | 0.0107 | 0.00582 |
| | | 20 | 0.5273 | 0.0032 | 0.0062 | 0.5268 | 0.0124 | 0.0052 | 0.5266 | 0.0125 | 0.00503 |
| | | 25 | 0.5280 | 0.0039 | 0.0041 | 0.5275 | 0.0134 | 0.0031 | 0.5287 | 0.0146 | 0.00355 |
| (4, 10) | 0.4928 | 10 | 0.5130 | 0.0202 | 0.0128 | 0.5088 | 0.0160 | 0.0041 | 0.5128 | 0.0200 | 0.00855 |
| | | 15 | 0.5024 | 0.0096 | 0.0076 | 0.5026 | 0.0098 | 0.0031 | 0.5074 | 0.0146 | 0.00621 |
| | | 20 | 0.5032 | 0.0104 | 0.0060 | 0.5012 | 0.0084 | 0.0028 | 0.5065 | 0.0137 | 0.00452 |
| | | 25 | 0.5002 | 0.0074 | 0.0038 | 0.5009 | 0.0081 | 0.0021 | 0.5121 | 0.0193 | 0.00326 |
| (7, 10) | 0.2564 | 10 | 0.2654 | 0.0090 | 0.0028 | 0.2575 | 0.0011 | 0.0014 | 0.2589 | 0.0025 | 0.00198 |
| | | 15 | 0.2615 | 0.0051 | 0.0017 | 0.2610 | 0.0046 | 0.0011 | 0.2615 | 0.0051 | 0.00132 |
| | | 20 | 0.2605 | 0.0041 | 0.0012 | 0.2608 | 0.0044 | 0.0009 | 0.2610 | 0.0046 | 0.00102 |
| | | 25 | 0.2568 | 0.0004 | 0.0009 | 0.2559 | -0.0005 | 0.0007 | 0.2565 | 0.0001 | 0.00084 |
| (9, 10) | 0.1872 | 10 | 0.1965 | 0.0093 | 0.0016 | 0.1947 | 0.0075 | 0.0058 | 0.1975 | 0.0103 | 0.00145 |
| | | 15 | 0.1944 | 0.0072 | 0.0007 | 0.1925 | 0.0053 | 0.0009 | 0.1927 | 0.0055 | 0.00082 |
| | | 20 | 0.1938 | 0.0066 | 0.0006 | 0.1916 | 0.0044 | 0.0005 | 0.1920 | 0.0048 | 0.00053 |
| | | 25 | 0.1925 | 0.0053 | 0.0005 | 0.1902 | 0.0030 | 0.0004 | 0.1906 | 0.0034 | 0.00037 |

**Table 3.** Estimates of $R_s^L$ when $\theta = 2$ for Gamma prior in L(cons/*k*-*n*: F) system

| | | | MLE | | | Bayes estimate | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $(k, n)$ | $R_s^L$ | $m$ | $\hat{R}_s^L$ | Bias | MSE | $[R_s^{L*}]_{Lin}$ | Bias | ER | $[R_s^{L*}]_{MC}$ | Bias | ER |
| (3, 10) | 0.4732 | 10 | 0.4936 | 0.0204 | 0.0123 | 0.4842 | 0.0110 | 0.0087 | 0.4879 | 0.0147 | 0.01096 |
| | | 15 | 0.4912 | 0.0180 | 0.0085 | 0.4816 | 0.0084 | 0.0065 | 0.4825 | 0.0093 | 0.00752 |
| | | 20 | 0.4865 | 0.0133 | 0.0061 | 0.4765 | 0.0033 | 0.0054 | 0.4772 | 0.0040 | 0.00568 |
| | | 25 | 0.4810 | 0.0078 | 0.0045 | 0.4772 | 0.0040 | 0.0012 | 0.4763 | 0.0031 | 0.00431 |
| (4, 10) | 0.3719 | 10 | 0.3812 | 0.0093 | 0.0081 | 0.3847 | 0.0128 | 0.0045 | 0.3877 | 0.0158 | 0.00145 |
| | | 15 | 0.3735 | 0.0016 | 0.0054 | 0.3849 | 0.0130 | 0.0039 | 0.3851 | 0.0132 | 0.00136 |
| | | 20 | 0.3724 | 0.0005 | 0.0021 | 0.3863 | 0.0144 | 0.0036 | 0.3868 | 0.0149 | 0.00121 |
| | | 25 | 0.3720 | 0.0001 | 0.0014 | 0.3884 | 0.0165 | 0.0028 | 0.3880 | 0.0161 | 0.00092 |
| (7, 10) | 0.2256 | 10 | 0.2346 | 0.0090 | 0.0023 | 0.2225 | -0.0031 | 0.0014 | 0.2341 | 0.0085 | 0.00197 |
| | | 15 | 0.2338 | 0.0082 | 0.0017 | 0.2214 | -0.0042 | 0.0011 | 0.2354 | 0.0098 | 0.00136 |
| | | 20 | 0.2329 | 0.0073 | 0.0012 | 0.2236 | -0.0020 | 0.0009 | 0.2365 | 0.0109 | 0.00108 |
| | | 25 | 0.2314 | 0.0058 | 0.0009 | 0.2231 | -0.0025 | 0.0007 | 0.2378 | 0.0122 | 0.00116 |
| (9, 10) | 0.1523 | 10 | 0.1618 | 0.0095 | 0.0016 | 0.1517 | -0.0006 | 0.0029 | 0.1621 | 0.0098 | 0.00156 |
| | | 15 | 0.1597 | 0.0074 | 0.0007 | 0.1526 | 0.0003 | 0.0016 | 0.1601 | 0.0078 | 0.00072 |
| | | 20 | 0.1574 | 0.0051 | 0.0006 | 0.1519 | -0.0004 | 0.0012 | 0.1578 | 0.0055 | 0.00075 |
| | | 25 | 0.1550 | 0.0027 | 0.0002 | 0.1532 | 0.0009 | 0.0024 | 0.1563 | 0.0040 | 0.00059 |

Additionally, Table 8 discusses the Bayes estimates of MTTF. The Bayes estimates for $R_s^L$ and $R_s^C$ are obtained using Lindley's approximation and the MCMC method, with an informative Gamma $(\alpha, \beta)$prior distribution. Specifically, the prior distribution Gamma $(\alpha, \beta)$ is set to Gamma (1,5). The MCMC Bayesian estimates are based on 10,000 sampling, namely, $N$ = 10,000. In each case, the interval level for the credible intervals is 95%. To mitigate the influence of the initial distribution, we discarded the first 9000 iterations, commonly referred to as burn-in. Using Gibbs sampling, we obtained Bayesian estimates along with credible intervals by employing 1,000 sample.

**Table 4.** Estimates of $R_S^C$ when $\theta = 0.5$ for Gamma prior in C(cons/$k$-$n$: F) system

| $(k, n)$ | $R_S^C$ | $m$ | MLE | | | Bayes estimate | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | $\hat{R}_S^C$ | Bias | MSE | $[R_S^{C*}]_{Lin}$ | Bias | ER | $[R_S^{C*}]_{MC}$ | Bias | ER |
| (3, 10) | 0.8256 | 10 | 0.8125 | -0.0131 | 0.0031 | 0.8115 | -0.0141 | 0.0021 | 0.8262 | 0.0006 | 0.00329 |
| | | 15 | 0.8162 | -0.0094 | 0.0023 | 0.8105 | -0.0151 | 0.0017 | 0.8270 | 0.0014 | 0.00245 |
| | | 20 | 0.8245 | -0.0011 | 0.0018 | 0.8232 | -0.0024 | 0.0015 | 0.8259 | 0.0003 | 0.00193 |
| | | 25 | 0.8233 | -0.0023 | 0.0015 | 0.8210 | -0.0046 | 0.0012 | 0.8278 | 0.0022 | 0.00156 |
| (4, 10) | 0.6521 | 10 | 0.6423 | -0.0098 | 0.0019 | 0.6528 | 0.0007 | 0.0032 | 0.6535 | 0.0014 | 0.00089 |
| | | 15 | 0.6433 | -0.0088 | 0.0006 | 0.6548 | 0.0027 | 0.0024 | 0.6559 | 0.0038 | 0.00062 |
| | | 20 | 0.6459 | -0.0062 | 0.0005 | 0.6572 | 0.0051 | 0.0019 | 0.6583 | 0.0062 | 0.00054 |
| | | 25 | 0.6534 | 0.0013 | 0.0003 | 0.6581 | 0.0060 | 0.0015 | 0.6592 | 0.0071 | 0.00047 |
| (7, 10) | 0.4025 | 10 | 0.4036 | 0.0011 | 0.0037 | 0.3958 | -0.0067 | 0.0020 | 0.4042 | 0.0017 | 0.00043 |
| | | 15 | 0.4047 | 0.0022 | 0.0026 | 0.3974 | -0.0051 | 0.0017 | 0.4065 | 0.0040 | 0.00033 |
| | | 20 | 0.4012 | -0.0013 | 0.0018 | 0.3980 | -0.0045 | 0.0013 | 0.4123 | 0.0098 | 0.00026 |
| | | 25 | 0.4005 | -0.0020 | 0.0013 | 0.4021 | -0.0004 | 0.0010 | 0.4132 | 0.0107 | 0.00022 |
| (9, 10) | 0.2431 | 10 | 0.2321 | -0.0110 | 0.0007 | 0.2319 | -0.0112 | 0.0010 | 0.2435 | 0.0004 | 0.00253 |
| | | 15 | 0.2332 | -0.0099 | 0.0005 | 0.2328 | -0.0103 | 0.0012 | 0.2430 | -0.0001 | 0.00216 |
| | | 20 | 0.2356 | -0.0075 | 0.0004 | 0.2345 | -0.0086 | 0.0004 | 0.2450 | 0.0019 | 0.00188 |
| | | 25 | 0.2415 | -0.0016 | 0.0002 | 0.2410 | -0.0021 | 0.0013 | 0.2448 | 0.0017 | 0.00208 |

**Table 5.** Estimates of $R_S^C$ when $\theta = 1.5$ for Gamma prior in C(cons/$k$-$n$: F) system

| $(k, n)$ | $R_S^C$ | $m$ | MLE | | | Bayes estimate | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | $\hat{R}_S^C$ | Bias | MSE | $[R_S^{C*}]_{Lin}$ | Bias | ER | $[R_S^{C*}]_{MC}$ | Bias | ER |
| (3, 10) | 0.7254 | 10 | 0.7221 | -0.0033 | 0.0039 | 0.7268 | 0.0014 | 0.0034 | 0.7271 | 0.0017 | 0.00375 |
| | | 15 | 0.7235 | -0.0019 | 0.0025 | 0.7259 | 0.0005 | 0.0021 | 0.7266 | 0.0012 | 0.00261 |
| | | 20 | 0.7187 | -0.0067 | 0.0014 | 0.7264 | 0.0010 | 0.0012 | 0.7268 | 0.0014 | 0.00192 |
| | | 25 | 0.7145 | -0.0109 | 0.0009 | 0.7288 | 0.0034 | 0.0008 | 0.7289 | 0.0035 | 0.00223 |
| (4, 10) | 0.5423 | 10 | 0.5438 | 0.0015 | 0.0023 | 0.5447 | 0.0024 | 0.0020 | 0.5450 | 0.0027 | 0.01096 |
| | | 15 | 0.5321 | -0.0102 | 0.0018 | 0.5427 | 0.0004 | 0.0017 | 0.5429 | 0.0006 | 0.00699 |
| | | 20 | 0.5312 | -0.0111 | 0.0014 | 0.5431 | 0.0008 | 0.0013 | 0.5440 | 0.0017 | 0.00541 |
| | | 25 | 0.5305 | -0.0118 | 0.0011 | 0.5429 | 0.0006 | 0.0010 | 0.5445 | 0.0022 | 0.00432 |
| (7, 10) | 0.2851 | 10 | 0.2811 | -0.0040 | 0.0104 | 0.2809 | -0.0042 | 0.0014 | 0.2856 | 0.0005 | 0.00304 |
| | | 15 | 0.2807 | -0.0044 | 0.0078 | 0.2817 | -0.0034 | 0.0013 | 0.2867 | 0.0016 | 0.00285 |
| | | 20 | 0.2745 | -0.0106 | 0.0061 | 0.2754 | -0.0097 | 0.0008 | 0.2874 | 0.0023 | 0.00197 |
| | | 25 | 0.2732 | -0.0119 | 0.0036 | 0.2746 | -0.0105 | 0.0006 | 0.2877 | 0.0026 | 0.00136 |
| (9, 10) | 0.1879 | 10 | 0.1785 | -0.0094 | 0.0016 | 0.1775 | -0.0104 | 0.0008 | 0.1885 | 0.0006 | 0.00415 |
| | | 15 | 0.1774 | -0.0105 | 0.0007 | 0.1765 | -0.0114 | 0.0005 | 0.1891 | 0.0012 | 0.00374 |
| | | 20 | 0.1765 | -0.0114 | 0.0006 | 0.1742 | -0.0137 | 0.0004 | 0.1881 | 0.0002 | 0.00230 |
| | | 25 | 0.1816 | -0.0063 | 0.0004 | 0.1810 | -0.0069 | 0.0002 | 0.1895 | 0.0016 | 0.00078 |

**Table 6.** Estimates of $R_s^C$ when $\theta = 2$ for Gamma prior in C(cons/*k*-*n*: F) system

| $(k, n)$ | $R_s^C$ | $m$ | MLE | | | Bayes estimate | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | $\hat{R}_s^C$ | Bias | MSE | $[R_s^{C*}]_{Lin}$ | Bias | ER | $[R_s^{C*}]_{MC}$ | Bias | ER |
| (3, 10) | 0.6458 | 10 | 0.6427 | -0.0031 | 0.0087 | 0.6321 | -0.0137 | 0.0034 | 0.6465 | 0.0007 | 0.00041 |
| | | 15 | 0.6420 | -0.0038 | 0.0060 | 0.6348 | -0.0110 | 0.0028 | 0.6460 | 0.0002 | 0.00032 |
| | | 20 | 0.6358 | -0.0100 | 0.0047 | 0.6351 | -0.0107 | 0.0025 | 0.6471 | 0.0013 | 0.00287 |
| | | 25 | 0.6345 | -0.0113 | 0.0037 | 0.6387 | -0.0071 | 0.0023 | 0.6476 | 0.0018 | 0.00249 |
| (4, 10) | 0.4825 | 10 | 0.4820 | -0.0005 | 0.0164 | 0.4729 | -0.0096 | 0.0040 | 0.4835 | 0.0010 | 0.00415 |
| | | 15 | 0.4812 | -0.0013 | 0.0123 | 0.4716 | -0.0109 | 0.0029 | 0.4841 | 0.0016 | 0.00367 |
| | | 20 | 0.4758 | -0.0067 | 0.0088 | 0.4810 | -0.0015 | 0.0025 | 0.4851 | 0.0026 | 0.00295 |
| | | 25 | 0.4769 | -0.0056 | 0.0080 | 0.4821 | -0.0004 | 0.0023 | 0.4859 | 0.0034 | 0.00203 |
| (7, 10) | 0.2165 | 10 | 0.2143 | -0.0022 | 0.0061 | 0.2158 | -0.0007 | 0.0049 | 0.2187 | 0.0022 | 0.00168 |
| | | 15 | 0.2154 | -0.0011 | 0.0038 | 0.2161 | -0.0004 | 0.0019 | 0.2212 | 0.0047 | 0.00145 |
| | | 20 | 0.2137 | -0.0028 | 0.0027 | 0.2078 | -0.0087 | 0.0010 | 0.2215 | 0.0050 | 0.00131 |
| | | 25 | 0.2162 | -0.0003 | 0.0021 | 0.2089 | -0.0076 | 0.0009 | 0.2231 | 0.0066 | 0.00119 |
| (9, 10) | 0.1639 | 10 | 0.1632 | -0.0007 | 0.0039 | 0.1625 | -0.0014 | 0.0053 | 0.1665 | 0.0026 | 0.00116 |
| | | 15 | 0.1575 | -0.0064 | 0.0022 | 0.1609 | -0.0030 | 0.0009 | 0.1671 | 0.0032 | 0.00099 |
| | | 20 | 0.1564 | -0.0075 | 0.0017 | 0.1559 | -0.0080 | 0.0006 | 0.1660 | 0.0021 | 0.00087 |
| | | 25 | 0.1592 | -0.0047 | 0.0013 | 0.1588 | -0.0051 | 0.0005 | 0.1675 | 0.0036 | 0.00075 |

By examining Tables 1- 6 we can observe a consistent pattern where the MSE, ERs, and biases of the estimates tend to decrease as the sample size increases in all the considered scenarios of L(cons/*k*-*n*: F) and C(cons/*k*-*n*: F) system. Also, it can be observed that with an increase in the value of the parameter $\theta$, the estimates derived from both classical and Bayesian approaches exhibit a decreasing trend. Additionally, our observations reveal that in terms of ERs, the Bayes estimates obtained using Lindley's approximation generally yields comparatively better results than the MCMC method in both cases. In contrast, we find that the estimates obtained through the ML method are superior to the Bayes estimates based on Lindley's approximation in L(cons/*k*-*n*: F) system. However, when considering the C(cons/*k*-*n*: F) system, we observe that Lindley's approximation yields better results in terms of estimates. Nevertheless, there are certain points where the values are either similar or lower. This intriguing phenomenon is visually depicted in Figs. 1 to 5 and 7 to 9.

**Table 7.** Estimates of Credible intervals (CRIs) for $R_S^L$ and $R_S^C$

| $(k, n)$ | $\theta$ | $m$ | L (cons/$k$-$n$: F) | C(cons/$k$-$n$: F) |
|---|---|---|---|---|
| (3, 10) | 0.5 | 10 | (0.2945,0.3650) | (0.2361,0.3241) |
| | | 15 | (0.2531,0.3364) | (0.2105,0.3112) |
| | | 20 | (0.2351,0.3125) | (0.2026,0.3071) |
| | | 25 | (0.2174,0.3046) | (0.2012,0.3035) |
| | 1.5 | 10 | (0.3066,0.3675) | (0.2851,0.3544) |
| | | 15 | (0.2968,0.3459) | (0.2729,0.3251) |
| | | 20 | (0.2645,0.3325) | (0.2538,0.3168) |
| | | 25 | (0.2492,0.3057) | (0.2415,0.3041) |
| | 2.0 | 10 | (0.3265,0.3817) | (0.3184,0.3742) |
| | | 15 | (0.3169,0.3772) | (0.3028,0.3661) |
| | | 20 | (0.2991,0.3657) | (0.2936,0.3604) |
| | | 25 | (0.2761,0.3550) | (0.2632,0.3518) |
| (4, 10) | 0.5 | 10 | (0.2133,0.2962) | (0.1965,0.2877) |
| | | 15 | (0.2054,0.2810) | (0.1844,0.2763) |
| | | 20 | (0.1884,0.2635) | (0.1756,0.2542) |
| | | 25 | (0.1625,0.2485) | (0.1526,0.2321) |
| | 1.5 | 10 | (0.2832,0.3247) | (0.2724,0.3136) |
| | | 15 | (0.2745,0.3084) | (0.2710,0.3021) |
| | | 20 | (0.2658,0.2978) | (0.2527,0.2963) |
| | | 25 | (0.2571,0.2866) | (0.2463,0.2818) |
| | 2.0 | 10 | (0.3049,0.3129) | (0.2947,0.3028) |
| | | 15 | (0.2851,0.3023) | (0.2793,0.2980) |
| | | 20 | (0.2337,0.2858) | (0.2205,0.2775) |
| | | 25 | (0.2267,0.2785) | (0.2245,0.2673) |
| (7, 10) | 0.5 | 10 | (0.1956,0.2146) | (0.1875,0.2033) |
| | | 15 | (0.1885,0.2024) | (0.1765,0.1990) |
| | | 20 | (0.1773,0.2007) | (0.1662,0.1986) |
| | | 25 | (0.1674,0.1965) | (0.1557,0.1874) |
| | 1.5 | 10 | (0.2257,0.2541) | (0.2157,0.2441) |
| | | 15 | (0.2021,0.2461) | (0.2036,0.2380) |
| | | 20 | (0.1968,0.2335) | (0.1882,0.2242) |
| | | 25 | (0.1979,0.2310) | (0.1812,0.2305) |
| | 2.0 | 10 | (0.2129,0.2446) | (0.2020,0.2418) |
| | | 15 | (0.2031,0.2416) | (0.1977,0.2383) |
| | | 20 | (0.1975,0.2339) | (0.1843,0.2240) |
| (9, 10) | 0.5 | 10 | (0.1635,0.1974) | (0.1563,0.1879) |
| | | 15 | (0.1542,0.1865) | (0.1497,0.1760) |
| | | 20 | (0.1478,0.1810) | (0.1365,0.1740) |
| | | 25 | (0.1336,0.1766) | (0.1321,0.1712) |
| | 1.5 | 10 | (0.1865,0.2033) | (0.1782,0.1947) |
| | | 15 | (0.1765,0.1936) | (0.1645,0.1821) |
| | | 20 | (0.1652,0.1866) | (0.1552,0.1813) |
| | | 25 | (0.1449,0.1685) | (0.1338,0.1576) |
| | 2.0 | 10 | (0.2025,0.2267) | (0.2014,0.2170) |
| | | 15 | (0.1997,0.2136) | (0.1880,0.2026) |
| | | 20 | (0.1836,0.2019) | (0.1779,0.2008) |
| | | 25 | (0.1747,0.1958) | (0.1665,0.1864) |

**Table 8.** Bayesian Estimates of mean time to failure (MTTF)

| $(k,n)$ | $\theta$ | $m$ | $\mu_L^*$ | $\mu_C^*$ |
|---|---|---|---|---|
| (3, 10) | 0.5 | 10 | 10215 | 11562 |
| | | 15 | 989 | 996 |
| | | 20 | 856 | 879 |
| | | 25 | 680 | 745 |
| | 1.5 | 10 | 9026 | 10025 |
| | | 15 | 2341 | 3235 |
| | | 20 | 1128 | 1238 |
| | | 25 | 909 | 865 |
| | 2.0 | 10 | 4518 | 5012 |
| | | 15 | 1326 | 1002 |
| | | 20 | 845 | 884 |
| | | 25 | 387 | 298 |
| (7, 10) | 0.5 | 10 | 2712 | 3035 |
| | | 15 | 1002 | 1147 |
| | | 20 | 742 | 663 |
| | | 25 | 245 | 269 |
| | 1.5 | 10 | 974 | 1010 |
| | | 15 | 810 | 878 |
| | | 20 | 315 | 341 |
| | | 25 | 224 | 316 |
| | 2.0 | 10 | 696 | 702 |
| | | 15 | 340 | 510 |
| | | 20 | 238 | 360 |
| | | 25 | 187 | 198 |
| (9, 10) | 0.5 | 10 | 552 | 789 |
| | | 15 | 502 | 663 |
| | | 20 | 445 | 386 |
| | | 25 | 299 | 348 |
| | 1.5 | 10 | 325 | 345 |
| | | 15 | 274 | 256 |
| | | 20 | 225 | 204 |
| | | 25 | 187 | 196 |
| | 2.0 | 10 | 268 | 314 |
| | | 15 | 156 | 165 |
| | | 20 | 148 | 152 |
| | | 25 | 139 | 141 |

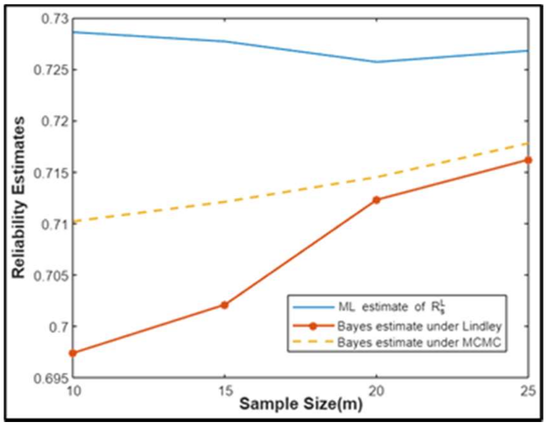


**Fig. 1:** Estimates of $R_S^L$ when $\theta$ =0.5 for L(cons/3-10) system

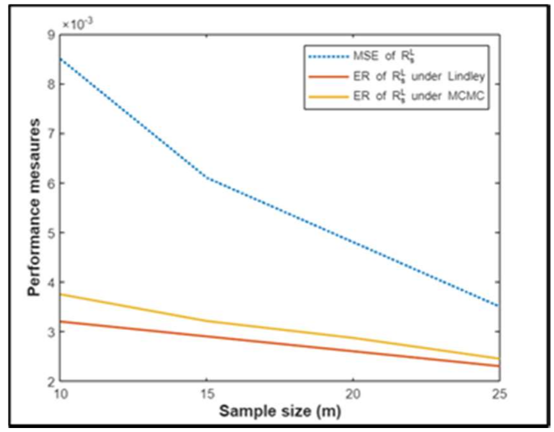


**Fig. 2:** MSEs and ERs of estimates of $R_S^L$ when $\theta$ =0.5 for L(cons/3-10) system

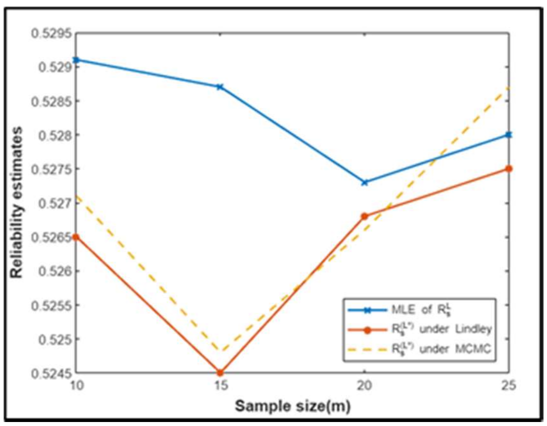


**Fig. 3:** Estimates of $R_S^L$ when $\theta$ = 1.5 for L(cons/3-10) system

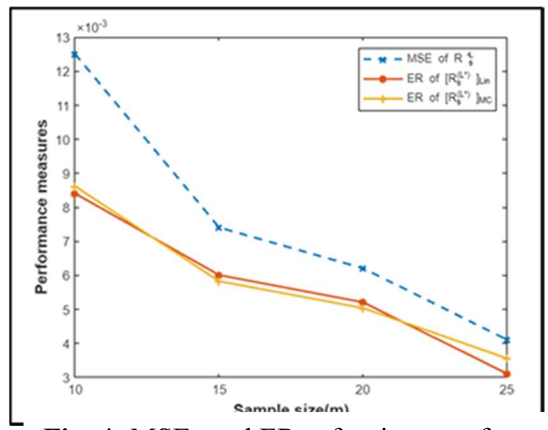


**Fig. 4:** MSEs and ERs of estimates of $R_S^L$ when $\theta$ =1.5 for L(cons/3-10) system

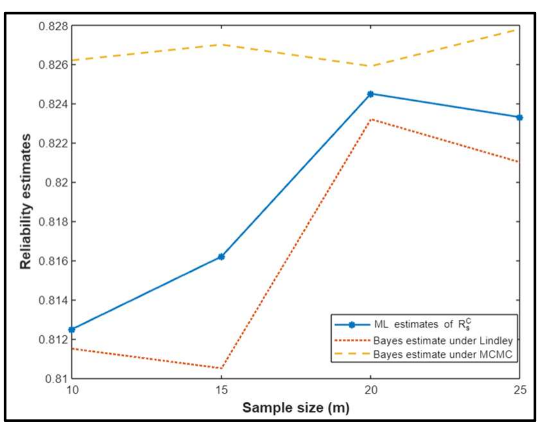


**Fig. 5:** Estimates of $R_S^C$ when $\theta$ = 0.5 for C(cons/3-10) system

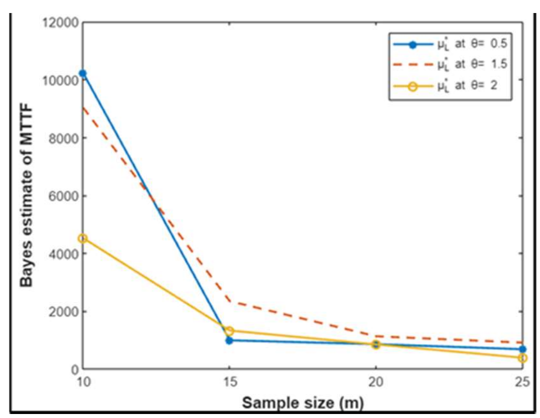


**Fig. 6:** Bayes estimates of MTTF at various values of $\theta$ for L(cons/3-10) system

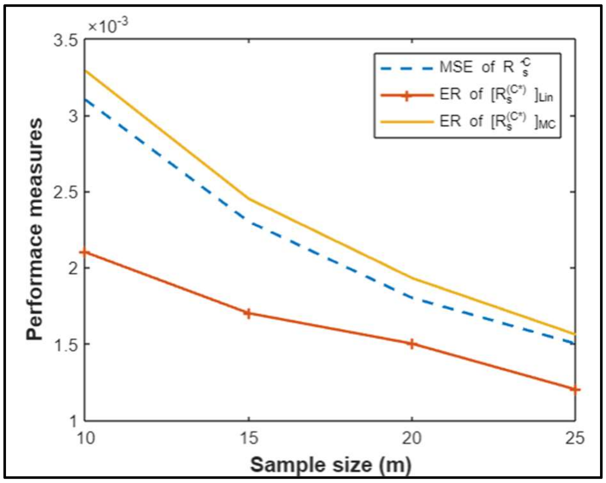


**Fig. 7:** MSEs and ERs of estimates of $R_S^C$ when $\theta$ =0.5 for C(cons/3-10) system

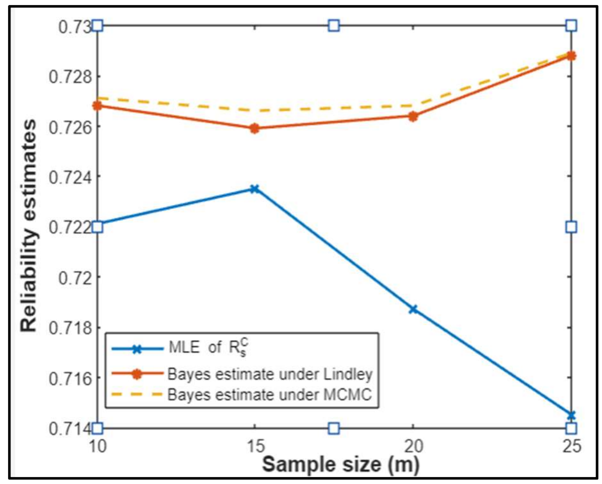


**Fig. 8:** Estimates of $R_S^C$ when $\theta$ = 1.5 for C(cons/3-10) system

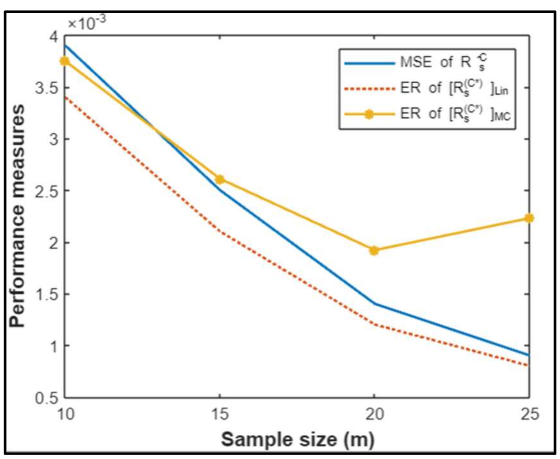


**Fig. 9:** MSEs and ERs of estimates of $R_S^C$ when $\theta$ =1.5 for C(cons/3-10) system

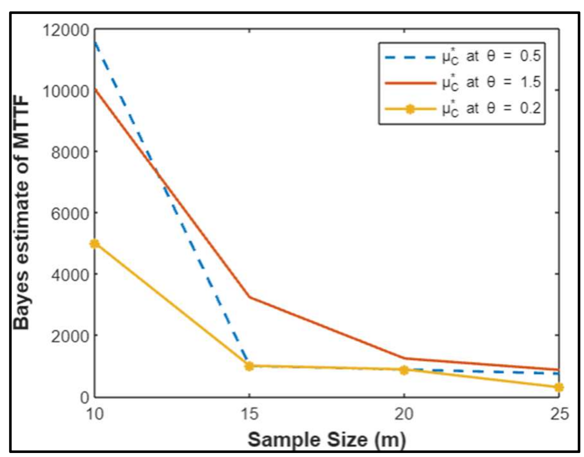


**Fig. 10:** Bayes estimates of MTTF at various values of $\theta$ for C(cons/3-10) system

According to the data in Table 7, the credible intervals tend to decrease as the sample size increases when using a Gamma prior distribution for different values of $\theta$. This trend holds true for both L(cons/*k*-*n*: F) and C(cons/*k*-*n*: F) systems. Moreover, when comparing the CRIs obtained from L(cons/k-n: F) and C(cons/*k*-*n*: F), it is observed that the CRIs from L(cons/*k*-*n*: F) are generally shorter in length than those from C(cons/*k*-*n*: F). This suggests that

L(cons/*k*-*n*: F) provides more precise estimates or predictions with a smaller range of plausible values for the parameter of interest.

Table 8 presents the Bayes estimates of MTTF, and from the tabulated values, it is evident that the estimates $\mu_L^*$ and $\mu_C^*$ decrease with increasing sample size across various values of $\theta$. Additionally, it is observed that as the value of $\theta$ decreases, the estimates $\mu_L^*$ and $\mu_C^*$ tend to increase. In summary, the data suggests that larger sample sizes result in lower estimates of $\mu_L^*$ and $\mu_C^*$, while smaller values of $\theta$ correspond to higher estimates of $\mu_L^*$ and $\mu_C^*$. These findings are visually presented in Figs. 6 and 10.

## VI. Conclusion

In this research paper, we investigated the estimation of consecutive linear/circular $k$-out-of-*n:* F system reliability using classical and Bayesian approaches, considering the Weighted Exponential-Lindley distribution as the lifetime distribution. In the classical estimation framework, we employed the ML method to obtain the ML estimators for the L(cons/*k*-*n*: F) and C(cons/*k*-*n*: F) system reliabilities. For Bayesian estimation, we utilized Lindley's approximation along with MCMC methods. We compared the ML and Bayesian estimates of system reliability in terms of biases and ERs. Additionally, we constructed Bayesian credible intervals for both L(cons/*k*-*n*: F) and C(cons/*k*-*n*: F) systems, providing a measure of uncertainty around the estimated reliabilities. Furthermore, we derived the Bayesian estimate of MTTF for all the considered cases of L(cons/*k*-*n*: F) and C(cons/*k*-*n*: F) systems. To assess the effectiveness of the proposed estimation methods, we conducted a Monte Carlo simulation study. It is observed that in terms of ERs, the Bayes estimates based on Lindley's approximation demonstrates better performance than the ML estimates and Bayes estimates based on MCMC method in all considered cases of both systems. However, The estimates based on ML method performs better in compare to Bayes estimates based on Lindley's approximation and MCMC method in all cases of L (cons/*k*-*n*: F) system. In contrast, Lindley's approximation yields better results in terms of estimates compare to rest of the estimation methods in most of the cases of C(cons/*k*-*n*: F) system. Moreover, we observed that the credible intervals for the C(cons/*k*-*n*: F) systems were consistently wider compared to those for the L(cons/*k*-*n*: F) system.

**Conflict of Interests**

The author(s) declare that there is no conflict of interests.

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