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Generalized Lauricella Function & Fractional Differential Operators Involving Multivariable H – Function

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Abstract- In this paper we use fractional differential operators and to derive a number of key formulas of multivariable H-function. We use the generalized Leibnitz's rule for fractional derivatives in order to obtain one of the aforementioned formulas, which involve a product of two multivariables H-function. It is further shown that ,each of these formulas yield interesting new formulas for certain multivariable hypergeometric function such as generalized Lauricella function (Srivastava-Dauost) and Lauriella hypergeometric function some of these application of the key formulas provide potentially useful generalization of known result in the theory of fractional calculus.

Keywords- Fractional differential operator, multivariable H-function.

I. INTRODUCTION

The fractional derivative of special function of one and more variables is important such as in the evaluation of series,[10,15] the derivation of generating function [12,chap.5] and the solution of differential equations [4,14;chap-3] motivated by these and many other avenues of applications, the fractional differential operators and are much used in the theory of special function of one and more variables.

We use the fractional derivative operator defined in the following manner [16]

$$D^{n}_{k,\alpha,x}(x^{\mu})=\prod_{r=0}^{n-1}[\frac{\sqrt{\mu+rk+1}}{\sqrt{\mu+rk-\alpha+1}}]\mathrm{x}^{\mu+nk} \quad (1)$$

Whereα≠μ+1 andα and k are not necessarily integers

We use the binomial expansion in the following manner

$$(ax^{\mu}+b)^{\lambda}=b^{\lambda}\sum_{l=0}^{\infty}\binom{\lambda}{l}\left(\frac{ax^{\mu}}{b}\right)^{l} \qquad where\ [\frac{ax^{\mu}}{b}]<1 \quad (2)$$

The familiar differential operator

Then it is known that the multiple Mellin-Barnes counter integral representing the multivariable H-function converges absolutely under the condition when

$$\xi_i=\min\left\{\mathrm{Re}\left(d_j^{(i)}/\delta_j^{(i)}\right)\right\}, \qquad (j=1,...,m_i)$$

$$\eta_i=\max\left\{\mathrm{Re}\left(c_j^{(i)}-1/\gamma_j^{(i)}\right)\right\}, \qquad (j=1,...,n_i) \quad (3)$$

The fractional calculus operators, involving various special function, has been found highly significant. It has gained popularity due to diverse application in fields, like physical sciences and engineering. During the past four decades, a number of workers have studied, in depth, the properties, applications, and different extensions of various operators of fractional calculus. A detailed account of such operators along with their properties and applications have been considered by several authors. and the closely related references therein.

Definition 1.2 (Marichev–Saigo–Maeda fractional differential operators [19]). Let $x > 0$ and $\alpha, \alpha', \beta, \beta', \gamma \epsilon C$ $such$ $that$ $R(\gamma) > 0$ and $k = [R(\gamma)] + 1$. Then

$$\left(I_{0,x}^{\alpha,\alpha',\beta,\beta',\gamma} f\right)(x) = \left(I_{0,x}^{-\alpha,-\alpha',-\beta',-\beta,-\gamma} f\right)(x)$$

$$= \left(\frac{d}{dx}\right)^{k} \left(I_{0,x}^{-\alpha,-\alpha',-\beta'+k,-\gamma+k} f\right)(x)$$

$$= \frac{1}{x^{\alpha}\Gamma(k-\gamma)}\left(\frac{d}{dx}\right)^{k} (x)^{\alpha'} \int_0^x (x-t)^{k-\gamma-1} \ t^{\alpha}$$

$$\times F_3\left(-\alpha', -\alpha. k-\beta; k-\gamma; 1-\frac{t}{x}, 1-\frac{x}{t}\right) f(t)dt \quad (4)$$

$$\left(I_{x,\infty}^{\alpha,\alpha',\beta,\beta',\gamma} f\right)(x) = \left(I_{x,\infty}^{-\alpha',-\alpha,-\beta',-\beta,-\gamma} f\right)(x)$$

$$= \left(-\frac{d}{dx}\right)^{k} \left(I_{x,\infty}^{-\alpha',-\alpha,-\beta'+k,-\beta,-\gamma+k} f\right)(x)$$

$$= \frac{1}{x^{\alpha}\Gamma(k-\gamma)}\left(-\frac{d}{dx}\right)^{k} (x)^{\alpha'} \int_0^x (t-x)^{k-\gamma-1} \ t^{\alpha'} \quad (5)$$

II. MAIN RESULT

Throughout the present paper .we assume that the convergence and existence condition corresponding appropriately to the ones detained above are satisfied by each of the various H-function involved in our results which are presented in the following sections

In this section we shall prove our main formulas on fractional differential operator involving multivariable H-function

Theorem-1

$$\Rightarrow D_{k,\alpha,x}^{n} D_y^{\mu} \{x^k y^{\lambda} (x^{v_1}+a)^{\sigma} (b-x^{v_2})^{-\delta} (y^{v_3}+c)^{h} (d-y^{v_4})^{-g} H[z_1 x^{k_1} y^{\lambda_1} \dots z_r x^{k_r} y^{\lambda_r}]$$

$$\min(\upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4, \sigma_i, \delta_i, h_i, g_i) > 0 \ (i = 1, \dots, \mathrm{r})$$

$$\max\left\{\left|\arg\left(x^{\upsilon_1}/a\right)\right|, \left|\arg\left(x^{\upsilon_2}/b\right)\right|, \arg\left|\left(y^{\upsilon_3}/c\right)\right|, \arg\left|\left(y^{\upsilon_4}/d\right)\right|\right\} < \pi$$

$$\mathrm{Re}(k) + \sum_{i=1}^{r} k_i \xi_i > -1 \ \mathrm{Re}(\lambda) + \sum_{i=1}^{r} \lambda_i \xi_i > -1$$

And where $\xi_1, \dots \xi_r$ are given in (1.3)

$$\Rightarrow a^{\sigma} b^{-\delta} c^{h} d^{-g} x^{k+nk} y^{\lambda-u} \sum_{l,m,r,s=0}^{\infty} \binom{\sigma}{l}\binom{\delta}{m}\binom{h}{r}\binom{g}{s}\left(\frac{x^{v_1}}{a}\right)^{l}\left(\frac{x^{v_2}}{b}\right)^{m}\left(\frac{y^{v_3}}{c}\right)^{r}\left(\frac{y^{v_4}}{d}\right)^{s}$$

$$H_{p+n'+1, \quad q+n'+1 \ p_1, q_1 \dots\dots, p_r, q_r}^{0, n+n'+1 \ ; \ m_1, n_1 \dots\dots, m_r, n_r}$$

$$\left[\begin{matrix} z_1 x^{k_1} y^{\lambda_1} \\ \cdot \\ \cdot \\ \cdot \\ z_r x^{k_r} y^{\lambda_r} \end{matrix}\right|^{(-k-kr-v_1 l-v_2 m : k_1 \dots\dots k_r)_{0,n-1} (-\lambda_1 - v_3 r + v_4 s + \lambda_1 \dots\dots \lambda_r), (aj\ \alpha_j^1 ; \dots\dots \alpha_j^{(r)})_{1,p}}_{(\alpha - rk - v_1 l - v_2 m : k_1 \dots\dots k_r)_{0,n-1} (-\lambda + \mu - v_3 r + v_4 s \lambda_1 \dots\dots + \lambda_r) (b_j ; \beta_j^1 \dots\dots \beta_j^{(r)})_{1,q}}$$

$$\left.\begin{matrix} : \left(c_j^1, r_j^1\right)_{1,p_1} \dots\dots \left(c_j^{(r)}, \ v_j^{(r)}\right)_{1,p_r} \\ \cdot \\ : \left(d_j^1, \delta_j^1\right)_{1,q_1} \dots\dots \left(d_j^{(r)}, \ \delta_j^{(r)}\right)_{1,q_r} \end{matrix}\right] \quad (6)$$

Theorem-2

$$\Rightarrow D_{k,\alpha,x}^{n} \ D_y^{\mu} \{x^k y^{\lambda} (x^{v_1}+a)^{\sigma} (b-x^{v_2})^{-\delta} (y^{v_3}+c)^{h} (d-y^{v_4})^{-g}$$

$$H[z_1 x^{k_1} y^{\lambda_1} (x^{v_1}+a)^{\sigma_1} (b-x^{v_2})^{-\delta_1} (y^{v_3}+c)^{h_1} (d-y^{v_4})^{-g_1} \dots \dots \dots$$

$$z_r x^{k_r} y^{\lambda_r} (x^{v_1}+a)^{\sigma_r} (b-x^{v_2})^{-\delta_r} (y^{v_3}+c)^{h_r} (d-y^{v_4})^{-g_r}]\}$$

Provided (in addition to the appropriate convergence and existence conditions mentioned with (2.1)

$$\Rightarrow a^{\sigma} b^{-\delta} c^{h} d^{-g} \ x^{k+nk} \ y^{\lambda-\mu} \sum_{l,m,r,s=0}^{\infty} \frac{\left(\frac{x^{v_1}}{a}\right)^{l}\left(\frac{x^{v_2}}{b}\right)^{m}\left(\frac{y^{v_3}}{c}\right)^{r}\left(\frac{y^{v_4}}{d}\right)^{s}}{\lfloor l \lfloor m \lfloor r \lfloor s}$$

$$H_{p+n+5, \ q'+n+5 \ : \ p_1, q_1 \dots\dots p_r, q_r}^{0, \ n'+n+5 \ \ : \ m_1, n_1 \dots\dots m_r, n_r} \left[\begin{matrix} z_1 x^{k_1} y^{\lambda_1} a^{\sigma_1} b^{-\delta_1} c^{h_1} d^{-g_1} \\ \vdots \\ z_r x^{k_r} y^{\lambda_r} a^{\sigma_r} b^{-\delta_r} c^{h_r} d^{-g_r} \end{matrix}\right|^{(-\sigma, \sigma_1 \dots\dots \sigma_r)(1-\delta-m; \delta_1 \dots\dots \delta_r), (-h ; h_1 \dots\dots h_r)}_{(-\sigma+l, \sigma_1, \dots\dots \sigma_r)(1-\delta; \delta_1, \dots\dots \delta_r), (-h+r ; h_1 \dots\dots h_r)}$$

$$\left.\begin{matrix} (1-g-s; g_1 \dots\dots g_r)(-\lambda - lv_3 - sv_4; \lambda_1 \dots\dots \lambda_r)(-k - lv_1 - mv_2; k_1 \dots\dots k_r)_{0,n-1} \left(a_j; \alpha_j^1 \dots\dots \alpha_j^{(r)}\right)_{1,p} \left(c_j^1 r_j^1\right)_{1,p_1} \dots \left(c_j^{r} r_j^{r}\right)_{1,p_r} \\ (1-g; g_1 \dots\dots g_r)(-\lambda + \mu - lv_3 - sv_4; \lambda_1 \dots\dots \lambda_r)(-k + n - lv_1 - mv_2; k_1 \dots\dots k_r)_{0,n-1} \left(b_j; \beta_j^1 \dots\dots \beta_j^{(r)}\right)_{1,q} \left(d_j^1 \delta_j^1\right)_{1,q_1} \dots \left(d_j^{(r)} \delta_j^{(r)}\right)_{1,q_r} \end{matrix}\right| \quad (7)$$

Theorem-3

$$\Rightarrow D_x^{\mu}\, D_y^{w}\left\{x^k y^{\lambda}\,(x^{v_1}+a)^{\sigma}\,(b-x^{v_2})^{-\delta}\,(y^{v_3}+c)^{h}(d-y^{v_4})^{-g}\right.$$
$$H\big[z_1 x^{k_1} y^{\lambda_1}\,(x^{v_1}+a)^{\sigma_1}\,(b-x^{v_2})^{-\delta_1}\,(y^{v_3}+c)^{h_1}(d-y^{v_4})^{-g_1}\ldots\ldots\ldots$$
$$\left. z_r x^{k_r} y^{\lambda_r}\,(x^{v_r}+a)^{\sigma_r}\,(b-x^{v_2})^{-\delta_r}\,(y^{v_3}+c)^{h_r}(d-y^{v_4})^{-g_r}\big]\right\}$$

Provided (in addition to the appropriate convergence and existence conditions that (2.1)

$$\Rightarrow a^{\sigma} b^{-\delta} c^{h} d^{-g}\, x^{k-\mu}\, y^{\lambda-w} \sum_{l,m,r,s=0}^{\infty} \frac{\left(\frac{x^{v_1}}{a}\right)^{l}\left(\frac{x^{v_2}}{b}\right)\left(\frac{x^{v_3}}{c}\right)^{r}\left(\frac{x^{v_4}}{d}\right)^{s}}{\lfloor l\, \lfloor m\, \lfloor r\, \lfloor s}$$

$$H^{0,e+6\,:\,m_1,n_1,\ldots,m_r,n_r}_{p+6,q+6:p_1,q_1,\ldots,p_r,q_r}\left[\begin{matrix} z_1 x^{k_1} y^{\lambda_1} a^{\sigma_1} b^{-\delta_1} c^{h_1} d^{-g_1} \\ \vdots \\ z_r x^{k_r} y^{\lambda_r} a^{\sigma_r} b^{-\delta_r} c^{h_r} d^{-g_r} \end{matrix}\right|\begin{matrix}(-\sigma;\sigma_1,\ldots,\sigma_r)(1-\delta-m;\sigma_1,\ldots,\sigma_r)(-h;h_1,\ldots,h_r) \\ \\ (-\sigma+l;\sigma_1,\ldots,\sigma_r)(1-\delta;\sigma_1,\ldots,\sigma_r)(-h+r;h_1,\ldots,h_r)\end{matrix}$$

$$\left.\begin{matrix}(1-g-s;g_1,\ldots,g_r)(-k-lv_1-mv_2;k_1,\ldots,k_r)(-\lambda-lv_3-sv_4;\lambda_1,\ldots,\lambda_r)(a_j;\alpha_j^{1},\ldots,\alpha_j^{(r)})_{1,p_1},\ldots,(c_j^{r},\gamma_j^{r})_{1,p_r} \\ \\ (1-g;g_1,\ldots,g_r)(-k+u-lv_1-mv_2;k_1,\ldots,k_r)(-\lambda+u-lv_3-sv_4;\lambda_1,\ldots,\lambda_r)(b_j;\beta_j^{1},\ldots,\beta_j^{(r)})_{1,q_1},\ldots,(d_j^{(r)},\delta_j^{(r)})_{1,q_r}\end{matrix}\right] \quad (8)$$

Proof:- (Result-1) We first replace the multivariable H-function occurring on the LHS by its Mellin-Barnes integrals

Collected the powers of x, y, $\left(x^{\upsilon_1}+a\right),\left(b-x^{\upsilon_2}\right),\left(y^{\upsilon_3}+c\right),\left(d-y^{\upsilon_4}\right)$ and apply the binomial

$$\left(x+\xi\right)^{\lambda}=\xi^{\lambda}\sum_{l=0}^{\infty}\binom{\lambda}{l}\left(\frac{x}{\xi}\right); \qquad \left|\frac{x}{\xi}\right|<1$$

We then apply the formula [7, p.67 eq.4.4.4]

$$D_x^{\mu}\left(x^{\lambda}\right)=\frac{\Gamma 1+\lambda}{\Gamma 1+\lambda-\mu}x^{\lambda-\mu}, \quad \left[\mathrm{Re}(\lambda)>-1\right]$$

And

$$D^{n}_{k,\alpha,x}\left(x^{\mu}\right)=\prod_{r=0}^{n-1}\left[\frac{\Gamma\mu+rk+1}{\Gamma\mu+rk-\alpha+1}\right]x^{\mu+nk}$$

Where $\alpha \neq \mu+1$ and α and k are not necessarily integers and interpret the resulting Mellin-Barnes contour integrals as a H-function of r-variables we shall arrive at (2.1)

Proof:- (Result-1I) Same as proof Result-I

Proof:- (Result-1II) :-We first replace the multivariable H-function occurring on the LHS by its Mellin-Barnes integrals

Collected the powers of x, y, $\left(x^{\upsilon_1}+a\right),\left(b-x^{\upsilon_2}\right),\left(y^{\upsilon_3}+c\right),\left(d-y^{\upsilon_4}\right)$ and apply the binomial

$$\left(x+\xi\right)^{\lambda}=\xi^{\lambda}\sum_{l=0}^{\infty}\binom{\lambda}{l}\left(\frac{x}{\xi}\right); \qquad \left|\frac{x}{\xi}\right|<1$$

We then apply the formula [7, p.67 eq.4.4.4]

$$D_x^{\mu}\left(x^{\lambda}\right)=\frac{\Gamma 1+\lambda}{\Gamma 1+\lambda-\mu}x^{\lambda-\mu}, \quad \left[\mathrm{Re}(\lambda)>-1\right]$$

And then again apply the above formula

Where $\alpha \neq \mu+1$ and α and k are not necessarily integers and interpret the resulting Mellin-Barnes contour integrals as a H-function of r-variables we shall arrive at (2.3)

III. CONCLUSION

In this paper we use fractional differential operators and to derive a number of key formulas of multivariable H-function. We use the generalized Leibnitz's rule for fractional derivatives in order to obtain one of the aforementioned formulas, which involve a product of three multivariables H-function. It is further shown that ,each of these formulas yield interesting new formulas for certain multivariable hyper geometric function such as generalized Lauricella function (Srivastava-Dauost)and Lauriella hyper geometric function some of these application of the key formulas provide potentially useful generalization of known result in the theory of fractional calculus.

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