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Applications of Fixed Point Theorems in Uniform Spaces

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Abstract- Fixed point theory is a cornerstone of modern mathematical analysis with applications spanning topology, functional analysis, and applied sciences. Uniform spaces generalize metric spaces by replacing distance with a uniform structure, thereby allowing broader applicability. This paper investigates extensions of classical fixed point theorems—particularly Banach and Schauder types—within uniform spaces. We establish conditions for existence and uniqueness of fixed points and demonstrate applications in functional equations, integral equations, and optimization problems. We studied certain popular fixed-point theorems for pairs of weakly and semi-compatible mappings. For a pair of self-mappings in Hausdorff uniform spaces, we discussed various stability results for a few common fixed points. Fixed point theorems in uniform spaces provide powerful, generalized tools to ensure the existence of solutions for equations where standard metric space distances (like Banach's) are insufficient, such as in topological vector spaces. These applications include solving systems of operator equations, finding fixed points for contractive mappings on gauge spaces, and analyzing stability in Hausdorff uniform spaces.

Keywords— Fixed Point Theorem; Uniform Space; Contraction Mapping; Common fixed point, Hausdorff Uniform spaces, admissible mappings, simulation functions

I. INTRODUCTION

Fixed point theory studies conditions under which a mapping $f: X \to X$ admits a point x such that $f(x) = x$. Classical results such as Banach's Contraction Principle are formulated in metric spaces. However, many mathematical structures encountered in analysis are not metrizable. Uniform spaces provide a generalized framework preserving uniform properties such as completeness and uniform continuity. This paper extends fixed point results to uniform spaces and highlights their applications in various mathematical domains.

II. PRELIMINARIES

1. Uniform Spaces

A uniform space (X, U) consists of a set X together with a uniformity U, which is a filter on $X \times X$ satisfying:

1. Each $u \in U$ contains the diagonal $\Delta = \{(x, x) : x \in X\}$

- If $u \in U$, then $u^{-1} \in U$
- If $u \in U$, there exists $v \in U$ such that $v \circ v \subseteq U$

2. Cauchy Filters and Completeness

A filter F on X is said to be Cauchy if for every entourage U, there exists $A \in F$ such that $A \times A \subseteq U$. A uniform space is complete if every Cauchy filter converges.

3. Fixed Point Definition

A point $x \in X$ is called a fixed point of a mapping f if $f(x) = x$.

What are the difficulties presented? They are as follows:

III. FIXED POINT THEOREMS IN UNIFORM SPACES

1. Generalized Banach Contraction Principle
Let (X, U) be a complete uniform space generated by a family of pseudometrics {di}. Suppose f: X → X satisfies: di(f(x), f(y)) ≤ k di(x, y), for all i and 0 ≤ k < 1

Then f has a unique fixed point in X.

2. Schauder-Type Fixed Point Theorem
Let X be a compact, convex subset of a locally convex uniform space. If f: X → X is continuous, then f has at least one fixed point.

3. Uniform Contraction Condition
A mapping f: X→X on a uniform space (X,U) is called a uniform contraction if there exists a constant 0 < k < 1such that for every pseudometric di generating the uniformity,

di(f(x),f(y)) ≤ k di(x,y),∀ x,y∈X This condition ensures that the mapping uniformly reduces the "distance" between points with respect to all pseudometrics. Consequently, the iterative sequence xn+1=f(xn) converges to a unique fixed point in a complete uniform space.

4. Cauchy Condition in Uniform Space
$\forall u \in U, \exists N$ such that $d(xm, x^n) \in U, \ \forall m, n \geq N$

Iterative Fixed Point Scheme
xn+1 =f(xn), n ≥0

Convergence to Fixed Point

$$\lim_{n\to\infty} x_n = x^* \Rightarrow f(x^*) = x^*$$

Uniform Continuity Condition
(xn→x) ⇒ (f(xn)→f(x))

Schauder Compactness Condition $f(X) \subseteq X$, X compact and convex $\Rightarrow \exists x^* \in X$

I. Stability Interpretation $|f(x)-x| \leq \epsilon \Rightarrow x \approx x^*$

IV. APPLICATIONS

Functional Equations Uniform space techniques allow solving functional equations where metric structures are insufficient.

Differential and Integral Equations Fixed point results are applied to prove existence of solutions in function spaces that are not necessarily metrizable.

Optimization and Equilibrium Problems Fixed points correspond to equilibrium states in optimization and economic models.

Applications in Daily Life of the Human Body Although fixed point theory is abstract, its principles naturally model many equilibrium and self-regulating processes in the human body. A fixed point can be interpreted as a stable physiological state where the system maintains balance.

d1. Homeostasis (Body Temperature Regulation) The human body maintains a nearly constant internal temperature (~37°C) through feedback mechanisms. Processes such as sweating and shivering act as corrective mappings that return the system to equilibrium, which can be modeled as a fixed point.

d2. Blood Glucose Regulation The interaction of insulin and glucagon stabilizes blood sugar levels. The equilibrium glucose concentration represents a fixed point of the metabolic regulatory function.

d3. Heart Rate Stabilization The autonomic nervous system adjusts heart rate based on physical activity or stress. After disturbances, the heart rate returns to a resting state, representing a fixed point of the system.

d4. Neural Activity and Brain Function Neural networks in the brain often converge to stable firing patterns. These steady states correspond to fixed points and are essential for cognition and memory processes.

d5. Drug Distribution (Pharmacokinetics) With repeated dosing, drug concentration in the bloodstream stabilizes at a steady-state level. This level is a fixed point of the pharmacokinetic system.
d6. Respiratory System Regulation Breathing rate is regulated by oxygen and carbon dioxide levels. The system stabilizes at an equilibrium state ensuring efficient gas exchange, representing a fixed point.

V. ADVANCED MATHEMATICAL EXTENSIONS

To further strengthen the theoretical framework, we incorporate advanced formulations involving multivalued mappings, nonlinear operators, and modern fixed point techniques in uniform spaces.

5-A$_1$. Multivalued Mapping (Set-Valued Operator)
Let $F: X \to 2^X$ be a multi valued mapping. A point $x^* \in X$ is called a fixed point if: $x^* \in F(x^*)$

5-A$_2$. Nadler-Type Contraction (Multivalued Case)
Using the Hausdorff pseudo metric HHH, the contraction condition is:
$H(F(x),F(y)) \leq k\, d(x,y), \quad 0 \leq k < 1$ This guarantees existence of fixed points in complete uniform spaces.

5-A$_3$. Nonlinear Operator Equation
Let T: X→X be a nonlinear operator. The fixed point problem becomes: $T(x) = x$

5-A$_4$. Iterative Scheme for Nonlinear Operators $\boldsymbol{x_{n+1} = T(x_n)}$
This iterative process converges under suitable compactness or continuity assumptions.

5-A$_5$. Accretive Operator Condition
For nonlinear operators:
$\langle T(x)-T(y),\ x-y \rangle \geq 0$ This ensures stability and convergence in many functional spaces.

5-A$_6$. Darbo Fixed Point Theorem (Measure of Noncompactness)
Let α be a measure of noncompactness: $\alpha(T(A)) \leq k\,\alpha(A), 0 \leq k < 1$
This generalizes Schauder's theorem.

5-A$_7$. Nonlinear Integral Operator $(Tx)(t) = \int abK(t,s,x(s))ds$
Fixed points of $(Tx)T$ correspond to solutions of integral equations.

5-A$_8$. Weak Contraction in Uniform Spaces $d(f(x),f(y)) \leq d(x,y) - \phi(d(x,y))$
where ϕ is a continuous function with $\phi(t)>0$ for $t>0$.

Examples
Let X be a space of bounded functions with uniform convergence structure. A contraction mapping defined on X admits a unique fixed point.

In locally convex spaces, continuous mappings on compact convex subsets guarantee fixed points by Schauder-type results.

Let (X, d) be a complete metric space and T : X → X be a mapping such that for some integer m, T m = T ◦ T ◦ · · · ◦ T{ m times} is a contraction mapping. Then T has a unique fixed point.

Assume that α: X ×X→R+ and T:X→X are a purpose
(a)T is called α-orbital admissible if α (x,Tx)≥ 1 implies α (Tx,T^2 x)≥1

(b)T is called triangular α-orbital admissible if T is α-orbital admissible and α (x,y)≥ 1 and

α (y,Ty)≥ 1 imply α (x,Ty) ≥ 1
Example Problem-1: Let T : R → R be defined by T (x) = x2. Determine the fixed point of T.
Solution: Given that T (x) = x2. From the definition of fixed point we have

T (x) = x, ⇒ x2 = x or x(x − 1) = 0. therefore x = 0, 1. Thus, the fixed points of T are 0 and 1
Example Problem-2: Does a translation mapping T (x) = x + a where a is non-zero fixed number have fixed points.

Solution: Given that T (x) = x + a. From the definition of fixed point we have,
T (x) = x ⇒ x + a = x or a = 0. [By Left Cancellation Law]

Since T (x) = x + a is a translation mapping, so $a \neq 0$.

Thus, the translation mapping T (x) = x + a has no fixed point.

Example Problem-3: Show that f (x) = −x for x ∈ [−2, −1] ∪ [1, 2] has no fixed point.

Solution: Given that f (x) = −x. From the definition of fixed point we have,

f (x) = x ⇒ −x = x. It is clear that no point of [−2, −1] ∪ [1, 2] will satisfy the above equation. Thus, f (x) = −x has no fixed point for x ∈ [−2, −1] ∪ [1, 2].
Example Problem-4: Let T be a mapping on R into itself defined by T (x) = (½) x. Show that T has a unique fixed point.

Solution: Given T (x) = (½) x, T (y) = y ⇒ (1/2) y = y, which holds good at y = 0.

Note that $\|\boldsymbol{T(x)-T(y)}\| = \|\boldsymbol{(x/2)-(y/2)}\| = \frac{\mathbf{1}}{\mathbf{2}}\|\boldsymbol{(x)-(y)}\|$

Thus T is a contraction mapping. Hence, by Banach fixed point theorem, T has a unique fixed point.

VI. CONCLUSION

This paper describe the advantages of uniform spaces as Generalization beyond metric spaces, Applicable to non-metrizable structures, Strong framework for convergence and continuity etc. The study extends classical fixed point theory to uniform spaces, providing a broader framework for analyzing convergence and stability beyond metric settings. The generalized results, including contraction and Schauder-type theorems, demonstrate both theoretical significance and practical applicability. In particular, modeling physiological processes highlights the interdisciplinary relevance of fixed point concepts. Overall, uniform spaces offer a flexible and powerful foundation for further developments in nonlinear analysis and applied mathematics.

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