

Analytical Solutions of Heat Transfer in Phase Change problems - Review Analysis

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Abstract - Transient heat transfer problems described by non-linear partial differential equations along with the moving interface conditions are special type of boundary value problems known as moving boundary problems or Stefan Problems. Freezing/melting problems are referred as Stefan problems, as these problems are first encountered by Physician Joseph Stefan and proposed a model for the polar ice-melting problem. The essential feature of a system undergoing phase change is that a moving interface exists separating two regions of different thermo-physical properties at which energy is absorbed or released, separating the two phases. The objective of this paper is to get mathematical understanding of the heat and mass transfer of the phase change problems with unknown free boundaries and solutions for different type of Stefan problems through research article review analysis. This study gives a clear picture on mathematical modelling of phase change problems with different solution techniques to quantify the process to predict the evolution of the temperature field in the material, the amount of energy used and stored, the interface location and thickness, the interface velocity, final time of freezing and analysis of phase change processes at the macroscopic level.

Key words - Moving Boundary problems, interface, boundary conditions, heat transfer.

I. INTRODUCTION

Heat transfer problems with a phase-change such as melting and freezing problems are studied extensively in the last century. This is due to their wide scientific, technological, and industrial applications. This paper conducts review analysis of different type of phase change problems described in different geometries, with different initial and moving boundary conditions, and the exact mathematical modelling of these problems. This review survey mainly focusses on, the mathematical treatment of these problems and different techniques to solve these non-linear partial differential equations completely. These studies can be classified based on the heat transfer mechanism that drives the phase change process. Heat transfer mechanism is an important aspect to design devices for engineers. The three different modes of heat transfer via conduction, convection and radiation can dominate as transport mechanisms or

simultaneous processes also play major role in designing the models.

Modelling solidification is a complex process, which is influenced by many parameters such as physical properties of the material, environmental conditions, the material medium etc. There are no certain ideal conditions and parameters that can be taken always. The review is carried for the best possible solutions that exists for these types of problems, which are relevant to the present situation. Wide varieties of problems are taken into study to get a clear understanding of the solutions to different type of Stefan problems. Analytical solutions for the given type of problems are obtained using similarity variables, perturbation methods, domain decomposition methods, quasi-steady state approximations etc.

Physical phenomenon involved in the study of the phase change problems, and the necessity of the study of these problems are important as the study of these problems involve in various engineering problems such as surface ice distribution in polar

regions, food processing technology, geo-thermal units etc. [1]

II. NEED OF THE STUDY

Heat transfer problems with a phase-change such as melting and freezing have been studied extensively in the last century. This is due to their wide scientific and technological and industrial applications. Different approaches to these problems, depending the geometry of the problem gives a wider angle to understand these problems. The study focuses on phase change problems of solidification that ranges from simple one-dimensional analytical models to multi-dimensional problems of phase change, to understand different analytical techniques to solve phase change problems associated to freezing/melting process with moving boundaries. This includes the models which exhibit explicit solutions to a class of one-phase and models of multi-phase Stefan problems. Different type of approaches to solve the heat equation both in the presence of forcing term and in the absence of forcing help to understand the mechanism more clearly.

Explicit analytical and semi-analytical solutions for phase change problems

Stefan problems are non-linear and hence analytical solutions have mathematical difficulties. In the literature, analytical solutions have been reported for one and two-dimensional geometries, and are comparatively difficult to solve these equations in higher dimensions. In addition, temperature in the medium is assumed uniform, with constant thermal properties [2]. Within these confines, a succession of models of increasingly complicated phase-change processes are presented. A review of a long bibliography on moving and free boundary problems for phase-change materials (PCM) for the heat equation are discussed in [3].

Study starts with the basic models with simple fixed temperatures at the boundaries and then relaxes the assumption of constant temperature at the boundary by allowing the temperature at the boundary as a function of time, with periodically oscillating temperature profile, thus bringing

temperature change effects into the picture. Papers that explain the mechanical effects of induced heat flux due to the temperature differences in the medium that are explicitly solvable are discussed in the study and examined the effect of non-dimensional parameters on the solution. More precise models, which include mechanical effects, but do not admit explicit solutions, are studied in the last subsection. Models with super cooling, thus relaxing the assumption that the phase change occurs at the fixed freezing temperature are also discussed to analyse the effect of super cooling on the movement of free boundary and the heat transfer phenomenon.

A number of analytic models for freezing of water are briefly reviewed that are classified based on the heat transfer mechanism. Neumann and Stefan in the 1800s have reported an analytical solution for these phase change problems. Lamé and Clapeyron [4] studied these boundary value problems as early as 1831. Classic work by Neumann in the 1860s [5], is the early model for the growth of sea ice. The diffusion equation, with a moving boundary between ice and ocean resulting from conservation of heat, is solved in one dimension. In this case, it is assumed that the boundary is sharp and heat transport from ocean to ice is negligible. The freezing interface is found to move as the square root of time.

The effect of salt transport is not considered in the study. These problems are named after Stefan because of the sequence of articles [6, 7] written by Stefan, which resulted from his study of the melting of the polar ice cap around 1890. Because of the presence of a free or moving boundary, analytical solutions of Stefan problems have remained very limited and usually not available. Goodman [8], Reynolds and Dolton [9], Gupta and Banik [10] have investigated approximate an analytical method that yield solutions of Stefan problems in simple closed forms.

Soloman et al. [11] presented a similarity solution for the one phase solidification problem. He considered a mushy region. The problem is solved for two moving boundaries one in solid phase and the other in the mushy region along with the temperature. The

classical Lamé-Clapeyron [4] solution is a particular case of the solution given by Solomon [11]. Adriana C. Briozzo et al. [12] discussed in his paper exact solutions for Stefan problems with source function. He discussed two different cases. In the first case, source function depends on the heat flux and in the second case, it depends on the temperature. One phase non-dimensional Stefan problem with two different control functions of the type $F(u_x(0,t),t) = (\lambda_0/\sqrt{t})u_x(0,t)$ and $F(u(0,t),t) = (\lambda_0/t)u(0,t)$ were chosen for the two different cases. Analytical solutions are obtained for the two cases by choosing similarity variables.

Olawanle. P. Layeni et al. [13], solved phase change problem through an elementary approach with a spatio-temporal exponential forcing term and in the absence of forcing term. He obtained more general results. Explicit solutions for a wider class of moving boundary problems with variable heat flux boundary conditions are discussed. By reducing the governing differential equations to an equivalent differential-difference equation (DDE), he generalized recent and standard results in the literature. Furthermore, this approach naturally admits the square root type evolution of moving fronts and its compatible fields. Domingo Alberto Tarzia [14], have presented an explicit solution of similarity type for a class of free boundary problem in a semi-infinite material.

Ice storage in rectangular units is widely used in air conditioning cool storage systems. Fang, G.Y et al. [15] examined solidification properties in a rectangular enclosure and obtained a closed form analytical solution for the one-dimensional Solidification problem. He considered, the transfer of heat within the material is purely by conduction, and neglected stagnant liquid phase. This model uses first order accuracy solutions to obtain second and higher order solutions. Linear and quadratic approximate temperature distributions were considered to obtain the analytical solutions for temperature distribution, position of the interface and the interface velocity. Effect of Stefan number on various process parameters is analyzed and found that Stefan number is directly proportional to the solidification thickness.

A Stefan problem of diffusion to study the concentration of solvent in glassy polymers and the time dependency of the solution is analyzed by Donald S. Cohen et al. [16]. Fick's equation for the diffusion $C_t = (DC_x)_x$ was considered to study the concentration $C(x,t)$ and stress $\sigma(x,t)$ with appropriate boundary conditions at the interface $S(t)$, where D is the diffusion of the medium. A quasi-steady state approximation was used to solve analyze the time history of the moving front. Various asymptotic expansions for concentration and stress were considered to study the short and long-time behavior of the solution.

An explicit solution is obtained by Natalia et al. [17], for a two-phase melting problem. He considered constant heat flux at the boundary and variable latent heat of fusion. The objective of this paper is to obtain sufficient conditions for a free boundary problem in order to have an explicit solution of a similarity type. Temperature distributions in both the regions along with the melt interface for a semi-infinite region are evaluated. The solution of the free boundary problem is obtained as a particular case of zero heat flux at $x = +\infty$ and zero initial temperature.

Analytical and numerical investigations are carried out by Vynnycky et al. [18] in a rectangular geometry for the solidification problem. Convection heat transfer mechanism is taken into account. One vertical wall of the enclosure is placed below the melting point and the other is set above the melting point. This work gives asymptotic analysis in terms of both Rayleigh number (Ra) and Stefan number (St) in the region of $Ra \geq 1$ and $St \leq 1$ which is different than the earlier analysis. It has proved a correlation for the thickness of the frozen layer as a function of time. Based on correlations the model simplifies to a first order ODEs to evaluate the thickness of the frozen region. Numerical solution is evaluated using finite element method. Analytical solution and numerical solution have good agreement for the two-dimensional solidification problem, except complications arise near the lower part of the enclosure. Analytical solution results indicate that the thickness of the frozen layer is overestimated. These results are not uniform for the changing time.

Sea ice growth is modelled by Mark J. Mc Guinness [19]. A simple model of heat transport and salt transport, with turbulence is solved analytically. The problem is discussed in two different cases. In the first case turbulence is induced by a constant friction velocity. In the second case turbulence is buoyancy-driven. The heat flux from the freezing interface to the air and the heat flux from the ocean to the freezing interface are calculated. The results are analysed for these fluxes. It is found that the flux ratios are constant within the ice thickness range 0.1-2 m. It is concluded that the reduction of the interface temperatures, ice growth and increase in oceanic heat flux is caused due to the salt.

Alexandrov et al. [20] has reported an explicit analytical solution for the problem of directional solidification. The results are related to the nonstationary dynamics of directional solidification of a three-component alloy H₂O-KNO₃-NaNO₃ with two moving boundaries of the phase change. One is the primary mushy layer in which phase change of component 'A' takes place and the other is the cotectic mushy layer, in which components A and B crystalizes. Corresponding heat and diffusion equations are formulated in the primary mushy layer and a linear approximation for the temperature as a function of coordinate x . Temperature in the solidified region and the two mushy regions has a linear profile whereas its behavior differs from a linear dependence in the liquid region. The reason is that the solute concentration in the liquid region increases as the growing solid phase displaces impurities in to the liquid region and hence the phase transition temperature in the liquid region increases.

XIE Zheng Hui et al. [21] studied changes in the soil depths due to the seasonal effects. The variation of depths of the frozen and thawed soils is reduced to a moving boundary problem. The frozen and thawed depth moving boundaries are governed by the Stefan condition at the interface. To find the total ice content, modelling of the problem is described with an independent mass balance equation. This explains the liquid water and the solid ice states in the soil. Also, the model is solved for the positions of the frost/thaw depths in soil. Numerical simulations

are carried using an adaptive mesh method for the moving boundary problem. The solution determines depths of the frozen and thawed soil, water content and temperature distribution. A periodic sinusoidal upper boundary condition for temperature is applied. The effect of the upper boundary condition is verified on the frost/thaw depths and soil temperature, by varying soil thickness, ground surface temperature, annual amplitude of ground surface temperature and thermal conductivity. The results show that the effect of the periodical sinusoidal upper boundary condition can simulate the depths of frozen and thaw zones reasonably with a periodical forcing.

One dimensional Phase field model for solidification was studied by Nelson et al. [22] in a closed domain in the presence of natural convection by combining asymptotic analysis. The problem is defined in a finite domain. In this method of phase-eld model an ordered parameter is introduced to identify the liquid and solid phases of the material by allotting a value for the ordered parameter in the phase equations. This model takes care of the two issues reported in the phase-eld model of Hariharan et al. [23]. These models are numerically elegant to solve and instead of tracking the interface position explicitly, the phase-eld function describes the solid and liquid phases of the medium. The unique objective of this study is to find the effect of the system parameters on the solution along with finding the parametric range of the validity of these parameters, which has not been addressed by [23]. Further the relationship between the two solutions resulting from a phase-field model to that from a sharp-interface model are analysed. For small Stefan values asymptotic analysis is carried to analyze the changes in the solution. Numerical results show good agreement with the analytical solution results. The results show agreement between the two models for the position of the solidification front and the temperature profiles in the solid and liquid phases. However, corrections are needed to interface location and temperature profiles as it is assumed that the thickness of the interface is non-zero. However, the magnitude of these corrections increases with the speed of the front as the corresponding increase in the latent heat release

also increases. Coupling between the order parameter and the temperature in the phase-field model eliminate the corrections to second order by selecting proper kinetic coefficient. We can notice that there is an agreement between the results of the phase-field model and the sharp interface model for the temperature profiles, except near the solidification front.

Mariela C. Olguin et al. [24] produced an analytical solution for the freezing problem for water content materials. Both heat and mass transfers are coupled to study the freezing mechanics for the two-phase solidification problem with vapor concentration. The region is divided into three parts separated by two moving fronts. One is the sublimation front and the other is the freezing front. Separate equations are formulated for each sub-region with the balance equations of energy and mass. Stefan condition is imposed at the interface for energy conservations. The equations are solved for the temperatures in each sub-regions and the two moving fronts positions at any time t .

The model is solved by semi - analytical method. Sublimation front, freezing front and sublimation temperature are related with the first derivatives of sublimation front and freezing front which are solved using explicit of finite difference method. It is found that the rate of advancement of dehydrated region is slower than the freezing front. To study the parametric range of the influencing parameters on the solution, possible values of external parameters were chosen appropriately. Numerical results were compared with the solution of Lamé Clapeyron [4] with very high heat transfer coefficient, to match to the constant surface temperature without ice sublimation. We can observe good agreement of both the results.

A moving boundary problem of solidification of Lava-lake to find the positions of the two moving interfaces is discussed in the paper by Ajay Manglik [25]. This represents a non-uniform solidification problem. A convective heat- transfer from the surface of lava-lake into the atmosphere and a conductive heat transfer in to the country rock from base is taken into account. Time dependent

boundary conditions are described at the contact with the country rock for the problem. Semi-analytical solution is obtained using Fourier series method where in Fourier spectral approach is used to obtain the solution in the spatial domain. Appropriate Fourier sine series expansion for the temperature distribution which satisfies the boundary conditions reduces the heat transfer equation in to a system of first order ODEs. Modified finite difference scheme is used in the time domain. Property of orthogonality is applied, to solve for position of the moving fronts and for Fourier coefficients. To circumvent the singularity at the initial time equations are modified for small time analysis. These results are compared with the available analytical solutions of simple cases.

To study shoreline movement, a shoreline model for the sedimentary ocean basin is developed by V.R. Voller [26]. This is a Stefan problem for melting, in which latent heat is treated as a linear function of space and only the active liquid phase is taken into count. The main focus of this paper is to understand the surface processes interact with changes in sea level. This problem differs with the classic Stefan problem of melting by adopting a fixed flux and variable latent heat at the boundary $x=0$. An analytical solution is obtained using the similarity variables. This paper focuses, to examine the dynamics of the shoreline movement in sedimentary basin during the inactivity of the tectonic plates. The similarity solution brings a square root dependence of shoreline position with time as in the case of classical Stefan problem of melting of ice.

Unidirectional solidification in a square cavity below a cooling wall, with time changing heating temperature conditions at the bottom wall is studied by S. Kimura [27]. The square section is filled with distilled water and assuming a quasi-steady state condition, the one-dimensional phase change problem is solved for the average solid layer thickness. In the model three non-dimensional parameters are introduced to study their effect on the process. The dimensionless parameter Biot number (Bi) depends on the frozen region thickness at steady state. Stefan number (Ste), which explains the temperature difference between the cooling

upper wall and the liquid temperature, and Stefan number (Ste) based on the heating bottom wall and the liquid temperature.

At the steady state, a perturbation solution is obtained for the one-dimensional problem. Oscillating heating boundary condition at the bottom layer induce a heat flux in to the liquid region and this phenomenon has many applications such as the molten liquid with high temperature, the walls of the vessel are coated with thin solidified layers to prevent the direct contact of the molten material and the walls of the vessel. Since solidification induces momentum and temperature reduction in the liquid part, the sustainability range of these parameters on the vessel wall are verified. Also, an environmental issue related to the freezing and melting of water and ice in lakes and oceans can be observed due to the thermal and momentum surges in the water. The analytic solution using perturbation method is validated against a two-dimensional numerical solution. The analytical results are in with good agreement of numerical results.

Exact solutions for the Stefan like problems are discussed by Sadoun [28], in which refined integral method is used to achieve the exact solution. Exact solutions are discussed for four different cases in which one is the inverse Stefan problem. One dimensional diffusion problem is solved where the kernel function is approximated using the initial temperature distribution, which is an important step of the refined integral method. The regular constant temperature boundary conditions on the moving boundary are replaced by more general boundary condition such that the temperature and the heat flux at the moving boundary are treated as functions of time. The refined integral equation is obtained by combining Goodman heat balance integral equation [8] and the double integration method. The purpose of this is to remove the heat flux term, which is not incorporated in the model. In the approximated temperature profile distribution initial values of the time dependent parameters $c_i, i=1,2,3$ are obtained by calculating the temperature $u(x,0)$ with the initial distribution. Later a functional relation is established for and in terms of and these are substituted in the refined integral equation which will reduce to a

first order ODE for . Numerical integration is performed to solve as a function of t consequently and are obtained as functions of 't'. Finally replacing these parameters in the temperature profile an exact solution is obtained. In each case, to select the form of approximation to substitute in the integral equation, the corresponding given initial distribution of the dependent variable is used. This technique has led to obtain the corresponding exact solution in each benchmark test example. It is proved that this method is simple to apply and very effective to solve one-dimensional nonlinear moving boundary problems.

The solution of the one-phase Stefan problem is presented by Sara Barathi [29] with temperature boundary specifications. The mathematical modelling is developed to obtain temperature distribution and the position of the moving interface, which is described by a function. The proposed method is based on a scheme introduced by Cannon [30]. The advantage of the proposed method is that obtain the interface position and temperature distribution in the form of continuous functions, instead of discrete form as in the case of classic Stefan problems. As a result of the current approach the sequence of approximations, convergent to the exact solution. This is an excellent method for solving moving and other physical problems.

Variational Iteration Method is used for the analytical treatment of the linear and nonlinear partial differential equations. The basic essence of this method is to construct a correction functional using linear and non-linear operators. Slota [31] applied the VIM to get the analytical solution for the Stefan problem. The problem in the known boundary is first reduced to system of PDE and then the VIM is applied. For the problem of one phase Stefan problem the curvilinear geometry is reduced to a rectangular geometry with the introduction of new set of appropriate variables, which will reduce the problem to system of PDE. By choosing correction functional for the set of PDE which consists of auxiliary functions and appropriate Lagrange multipliers, the problem is solved to find the unknown moving boundary and the temperature. Finally, Iteration formulas are obtained for the

temperature distribution, and the position of the moving interface. The validity of the VIM is compared with the results obtained by an analytical solution.

Jafari et al. [32] applied the Homotopy Analysis Method (HAM) to solve one phase Stefan problem. This method has an advantage over numerical methods, because numerical methods use discretize the derivatives in the differential equations which will converge to the solution depending on the increments and develop lack of accuracy due to rounding of the errors. The HAM method is better as it doesn't involve discretization of the variables. The auxiliary linear operator, the initial guess, the auxiliary parameter, and the auxiliary function should be properly chosen, Taylor's series expansion of the unknown function converges. The validity of the method is based on an assumption that the series converges and the analytical solution by Homotopy analysis method is validated against Adomian decomposition method, exact solution for different values of the auxiliary parameter. Good agreement can be seen between the results.

Kushwaha et al. [33] has obtained an analytical solution for a one phase moving boundary problem using Adomian Decomposition Method (ADM) with variable latent heat. ADM method is applied to find the height of the sediment and position of the moving boundary problem. ADM method decomposes the unknown function in to an infinite series by applying an inverse operator for the given differential equation which is expressed in the operator form, such that the height of the shore line is expressed as decomposed terms. This method shows good agreement with the exact solution and also simple to solve non-linear problems that arise in the field of science and engineering.

Many diverse phenomena are non-linear behavior in nature. Perturbation methods are widely used by researchers to solve these behavior problems. The method basically involves a small parameter in the equation that explains the perturbations of the process. Suitable choice of small parameters leads to good results. Rajeev [34] obtained an approximate analytical solution for the Stefan problem with

variable latent heat by adopting Homotopy Perturbation method. Perturbation parameter is both implicit and explicit parameter. The modelling of the Shoreline in a sedimentary ocean basin is reduced to a Stefan problem with variable latent heat.

The problem is solved for the movement of the shoreline using HPM. This Shoreline Stefan problem involves propagation of shoreline which is caused due to sediment flux, tectonic subsidence of the earth's crust, and variations in sea level. Governing differential equation is the diffusion equation that models the transport of the sediment into the river basin. Results show that if the sediment line flux increases, the movement of the shoreline position increases towards sea. The results are compared with the analytical solution given by [26]. This model proves the accuracy of the HPM to solve non-linear problems.

Jitendra Singh et al. [35] reported a melting / freezing problem in a semi-infinite region for a two-phase Stefan problem, with the boundary condition of second kind using Vibrational Iteration Method (VIM). It is assumed that the physical properties, thermal conductivity and the specific heat depend on temperature. Temperature in both the regions is obtained and hence it is treated as two-phase problem. The advantage of this method over other methods is that the solution is obtained as continuous function, for the interface position and temperature distribution. Also, this method avoids linearization and the solution is analytic. Two cases have been discussed in the problem. Thermal conductivity and the specific heat are treated as function of temperatures in two regions. In the first case an exponential approximation is chosen for thermal conductivity and specific heat and in the second case a linear approximation is chosen for thermal conductivity and specific heat. The dimensionless interface position is plotted as a function of Stefan number for both the cases. A good agreement between the applied VIM and an exact solution can be seen.

Vibrational Iteration Method (VIM) is a powerful tool for solving wide range of Stefan like problems. In

paper [31], an approximate solution of one-phase Stefan problem by using VIM is presented. D. Słota et al. [36] reported an inverse Stefan problem assuming temperature boundary conditions of first kind. In this method unknown function is described as linear combination of given functions. Further using the concept of VIM, the minimal value of assumed function is found. Analytical solution of the Solidification problem of one phase, with boundary conditions of the second kind is given by Hetmaniok et al. [37] using VIM.

The proposed method reduces the problem into a system of ordinary differential equations and further for the given system the VIM is applied. In this method first the problem formulated for the curvy linear domain is reduced to a rectangular domain by using some proper transformations and then the VIM method is applied. That is a new correction functionals with Lagrange multipliers are introduced. This gives an iteration formula for the temperature distribution and moving boundary position. In the third step initial approximation for the temperature distribution is taken as a linear combination of the known functions that satisfy the modified initial and boundary conditions. Successive approximations converge to the exact solution.

Explicit solution for a moving boundary problem of freezing of moisture in a porous medium is discussed in the paper by Eduardo Adrian Santillan Marcus et al. [38]. Along with an exact solution for temperature and moisture distribution an equivalence between two boundary conditions at $x=0$ is brought out. In this study phase change problem with heat flux condition on the surface of the form and a fixed temperature are studied. In the problem with fixed temperature, the condition of latent heat is analyzed. The effect of the parameter on the movement of the interphase position, and for different values of , temperature distribution is studied. Finally, an inequality in terms of the coefficient between two different cases of the moving boundary problem is established.

Amir Gholaminejad [39] investigated water super cooling experimentally and the existence of this condition for super cooling. The main objective of

the study is to find the parameters that influence super cooling. Super cooling in an unstable phenomenon and occurs at special conditions. Usually when water reaches its freezing temperature (), freezing process starts without change in temperature. In super cooling, freezing does not sets in even though the temperature is below the freezing temperature. This phenomenon has more applications in food industry. It is shown in the study that super cooling takes place with a freezer temperature range of -3°C to -8°C and is independent of the initial temperature. When the freezer temperature decreases further the probability of nucleation process increases which in turn causes instant freezing. Also, he discussed the M-Pemba effect which states that initially hot water freezes faster than initially cold water which opposes the Newton's law of cooling. He pointed that hot water shows instability to super cooling.

Research on freezing and its applications are briefly reviewed by Saad Aldousari [40] and Akyurt et al. [41]. Potential practical uses of pressures generated by freezing of water and blockage are mentioned. During freezing internal pressure in the pipe increases at some stage the pressure in the pipe reaches to about 4000 psi and that leads to a burst and deformation of the pipe. Design concepts are introduced for doing work by utilizing pressures caused by ice blockage. Ice blocks are used to plug the pipes without draining the pipes at the time of maintenance work by engineers. Another commercial application of freezing is ground freezing, undoubtedly the most effective and consistently reliable method of providing temporary support and of preventing groundwater from owing into deep excavations. An emerging technic of thermal storage is the concept of storage of huge latent heat release at the time of freezing for shifting loads. Water is converted into ice when the demand for electricity is low, and the ice is melted during periods of peak demand to help with cooling and air conditioning loads.

Freezing of stagnant water in cylindrical geometry attracted many researchers due to the blockage and bursting of pipes. Mathematical modelling and simulation of the freezing time of water in small

diameter pipes is discussed in the paper by Andre McDonald et al. [42]. Transient conduction model in a one-dimensional space is developed to predict the freezing time was developed and validated against the experimental results. Freezing of water in cylindrical pipes conducted for stagnant and varying surrounding air temperatures. The model is solved numerically using separation of variables method for a finite length scale. For different inner diameters, results are drawn for cooling and freezing effects of water in copper and steel pipes. Numerical results are in good agreement with experimental results. Slight variations are observed in cooling times due to the assumption of constant surrounding temperature during the cooling phase of water.

Water transport to the interface during solidification process in unsaturated granular packed bed is studied experimentally and numerically by Rattanadecho [43]. Limited research has been reported in the field of heat transfer in unsaturated porous media and reported papers discuss only the solidification in porous media without an unsaturated state. The problem is one-dimensional and is solved using co-ordinate transformation technique based on a boundary fixing method coupled with implicit time schemes. Finite difference method based on the control volumes is used to solve the governing differential equation. Problem is solved numerically and experimentally for the temperature in the frozen region, position of the moving boundary and the water transport to the changing interface. Results show that the water transport depends on the heat flux at the boundary and the water saturation at the interface. This has an influence on the ice content in the frozen layer.

Calculus of variation is an excellent technique used to solve many engineering and physical problems related to phase change. Basirzadeh et al. [44] introduced this approach for solving moving boundary problems and for optimization. The method includes transforming the problem into an optimal control problem, by defining an objective function and auxiliary control functions. Then the non-linear problem is converted into linear form in which the goal is the minimization of a linear form over a subset of Radon measures. The graphs of the

piecewise constant control functions and the trajectory functions were drawn at different times. This theory of calculus of variation is proved to be a simple and efficient technique to solve the non-linear phase change problems.

III. CONCLUSION

It is observed that the available solutions that are considered for the review of the moving boundary problems partially represent the physical systems that study the heat transfer processes. There are very few models that consider the sublimation mechanism, to study the vapor concentrations in the dehydrated regions. Solidification with sublimation has been reported by [24] with unknown upper boundary sublimation temperature at the upper boundary of the freezing medium. In the above work, only the effect of heat transfer coefficient and surrounding temperatures considered for the discussions. Except this work, no similar models for heat and mass transfer with surface ice sublimation have been reported in the literature.

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