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Token Based Reconfiguration in Graph Theory: Complexity, Connectivity, and Applications

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Abstract- The reconfiguration framework provides a unified perspective on a wide range of combinatorial and graph-theoretic problems by modeling transformations between feasible solutions under defined adjacency rules. Using the classical 15-puzzle as a motivating example, the framework extends to source problems such as independent set, dominating set, graph coloring, satisfiability, and degree-sequence realizations. Central computational challenges—including reachability, shortest transformation sequences, connectivity, and diameter—are examined, many of which are PSPACE-complete in general but admit efficient algorithms for restricted graph classes. Token-based dynamics such as sliding, jumping, and addition-removal serve as fundamental adjacency models, while extensions to puzzles, coloring, and SAT reconfiguration highlight structural and complexity landscapes. This study emphasizes the deep connections between discrete mathematics and practical applications in physics, robotics, and algorithmic game theory, offering a comprehensive foundation for future research in reconfiguration problems[1,2].

Keywords: Reconfiguration, Graph Coloring, independent set, dominating set, token sliding, PSPACE completeness.

I. INTRODUCTION

The reconfiguration framework, although formally introduced recently, regroups a lot of mathematical problems that have been studied for decades in combinatorics and graph theory[1]. The questions raised in reconfiguration are natural and have concrete applications in fields such as physics, robotics, and popular games. A classic example is the 15-puzzle. This popular puzzle consists of 15 tiles arranged in a 4 × 4 grid with one hole; a move slides a tile adjacent to the hole into the hole[6]. In the mathematical model we consider 15 labeled tokens on the vertices v1, . . . , v16 of the 4 × 4 grid graph; the empty vertex is the one without a token. A move slides a token along an edge to the empty vertex. The goal is to find a sequence of moves from the current configuration to the configuration where token i is on vertex vi.

As underlined in the source, a reconfiguration problem is defined by five features: the source problem, the instance, the feasible solutions, the adjacency rule, and the problematic[2]. The source problem provides the set of feasible solutions; the instance specifies the input; the adjacency rule defines allowed moves; and the problematic is the computational or structural question (reachability, connectivity, diameter, etc.). Examples of source problems include independent set[8], dominating set[19], matching, graph coloring[11], and satisfiability[16].

Source problem, instance and feasible solutions

The aim of reconfiguration is to transform a solution of a source problem into another. The solutions we reconfigure are the solutions of the source problem. For example, in the 15-puzzle the instance is the 4 × 4 grid and the feasible solutions are the token placements. In independent set reconfiguration, feasible solutions are independent sets of a graph. This paper reviews many such source problems and their reconfiguration variants.

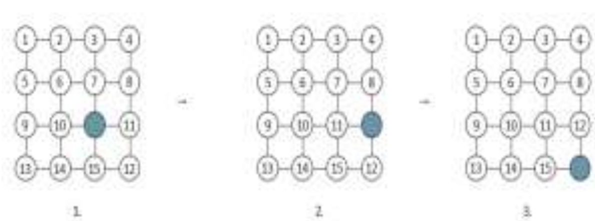


Figure 1: A sequence that solves a 15-puzzle in 2 steps.

Adjacency rule

The adjacency rule describes the single-step changes allowed between feasible solutions. In the 15-puzzle the move is sliding a token along an edge to the empty vertex [6]. In satisfiability reconfiguration a move flips the value of exactly one Boolean variable [16]. In degree-sequence reconfiguration a move flips two disjoint edges by exchanging endpoints [12,13]. Most works focus on mutual (symmetric) adjacency rules [1,2].

RECONFIGURATION GRAPH

Let Xs and Xt be two feasible solutions of a source problem Π. A reconfiguration sequence from Xs to Xt is a sequence S = ⟨X1 := Xs, . . . , Xℓ := Xt⟩ where each Xi is feasible and consecutive solutions differ by a single allowed move [1]. The reconfiguration graph R has vertices equal to feasible solutions and edges between solutions that differ by one move. Solving a 15-puzzle is equivalent to finding a path in R from the initial to the target configuration [6].

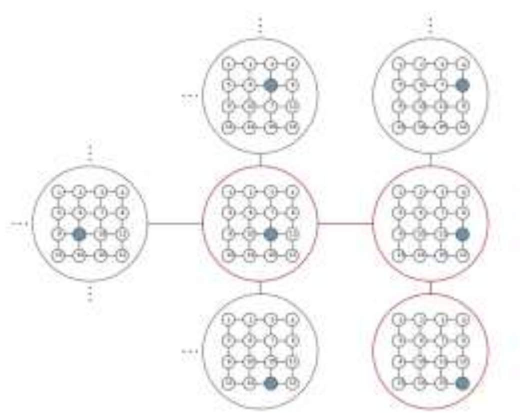


Figure 2: A portion of the reconfiguration graph Y for the 15-puzzle

Problematic

Typical problematics include:
[leftmargin=*]

Reachability: Given Π, instance I, and feasible solutions Xs, Xt, is there a reconfiguration sequence from Xs to Xt? (Decision problem; often PSPACE-complete.)

Shortest transformation sequence: Given reachable Xs, Xt, what is the minimum number of moves? A trivial lower bound uses the symmetric difference Δ(Xs, Xt).

Connectivity: Is R connected for a given instance?

Diameter: What is the maximum length of a shortest transformation between any two feasible solutions?

These problematics are central in this paper and recur in the subsequent sections.

Reconfiguration of Tokens

Many reconfiguration problems involve token positions on a graph G = (V, E). For independent set reconfiguration, feasible solutions are independent sets represented by tokens on vertices.

Adjacency rules

We review common token adjacency rules [9].

Token Sliding (TS): A move slides a token along an edge [9]. For independent sets, I and I′ are adjacent if there exist u, v ∈ V with uv ∈ E and I′ = (I ∪ {v}) \ {u}, denoted uv. The number of tokens remains constant; we study Rk where feasible solutions have size k.

Token Jumping (TJ): A move moves a token to any vertex. For independent sets, I′ is obtained by replacing u with v without requiring uv ∈ E. TJ is more permissive than TS [8].

Token Addition-Removal (TAR): A move adds or removes a token. TAR changes the number of tokens; we often restrict feasible solutions to size at least k (for maximization problems) or at most k (for minimization problems). The threshold k is a central parameter in many connectivity results [19,20].

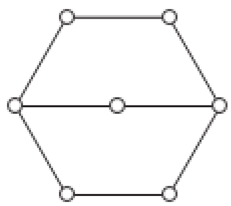
Generalizations of the 15-puzzle

Early results showed the 15-puzzle reconfiguration graph is not connected. For p × p grids the reconfiguration graph is disconnected; diameter bounds and complexity results follow. On general graphs, Wilson (1974) characterized connectivity:

Theorem. [Wilson, 1974] Consider the generalization of the 15-puzzle played on any graph G. The reconfiguration graph R is connected if and only if G is 2-connected, and it is not one of the following exceptions: bipartite graphs; cycles of order n ≥ 4; and a specific small exceptional graph.

If G is disconnected then R is disconnected; if G is 1-connected but not 2-connected there exist articulation vertices that prevent tokens from changing certain components. For cycles only cyclic permutations are possible, yielding many components. Diameter and complexity bounds for these generalizations are known (e.g., diameter O(n3) in some settings; shortest transformation NP-complete in others [18].

- bipartite graphs
- cycles of order $n \geq 4$



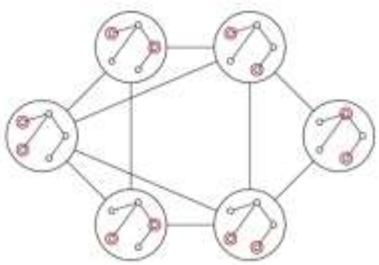


Figure3: The reconfiguration graph of γ3 for the Dominating Set reconfiguration problem under TS.

Independent Set and Vertex Cover Reconfiguration

Kaminski et al. proved TJ and TAR are equivalent for independent [8] reconfiguration: there exists a TJ sequence between two k-token configurations iff there exists a TAR sequence where feasible solutions have size at least $k - 1$. Under TJ, reachability is PSPACE-complete even for planar graphs [9] with maximum degree at most 3; under TS, reachability is also PSPACE-complete. The shortest transformation problem is PSPACE-complete in general, with polynomial-time algorithms for special graph classes.

Dominating Set Reconfiguration

Dominating Set reconfiguration has been studied mainly under TAR, with reachability PSPACE-complete even for split and bipartite graphs [19]; linear-time algorithms exist for trees and interval graphs. The connectivity threshold d0 (smallest k such that Rk is connected for all larger k) has been investigated: constructions show RΓ(G)+1 can be disconnected, and families exist with d0 = 2Γ(G) − 1. Positive connectivity results give sufficient thresholds guaranteeing connectivity and linear diameter in some cases. Recent work also studies TS for dominating sets, proving PSPACE-completeness for DSRTS on several graph classes while giving polynomial algorithms for cographs and dually chordal graphs [20,21].

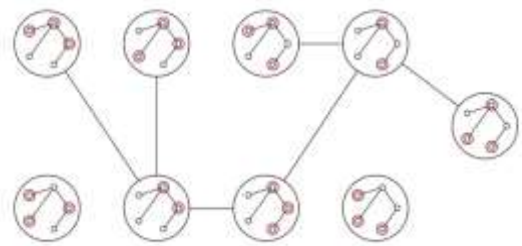


Figure 4: The reconfiguration graph R3 for Dominating Set under TAR.

II. FURTHER RECONFIGURATION PROBLEMS

Coloring Reconfiguration

Coloring reconfiguration studies proper k-colorings of a graph G; the re-configuration graph is Rk. Two main dynamics [16,17] are studied: Glauber dynamics (single-vertex recoloring) and Kempe dynamics (exchanging colors along Kempe chains). Connectivity and mixing properties relate to degeneracy deg(G). If k ≥ deg(G) + 2 then G is k-mixing; Cereceda conjectured that if k ≥ deg(G) + 2 then the diameter of Rk is O(n2). Recent progress gives polynomial diameter bounds under related conditions; many special graph classes satisfy quadratic or better diameter bounds.

Satisfiability Reconfiguration

For a CNF formula F, feasible solutions are satisfying assignments and moves flip one variable. SATR is PSPACE-complete [16, 17]; dichotomy and trichotomy results classify reachability and shortest transformation complexity based on relation classes

(Schaefer relations, tight relations [16], navigable relations). The shortest transformation problem is in P for navigable relations, NP-complete for tight but not navigable relations, and PSPACE-complete otherwise.

Reconfiguration of graphs with the same degree sequence

Feasible solutions are multigraphs realizing a given degree sequence [15]. A flip [12,13] replaces disjoint edges ab, cd by ac, bd (or ad, bc), preserving degrees. Hakimi and Taylor proved connectivity of the reconfiguration graph under flips for various graph families. Will gave an exact formula for the length of a shortest flip sequence between two multigraphs G, H with the same degree sequence:

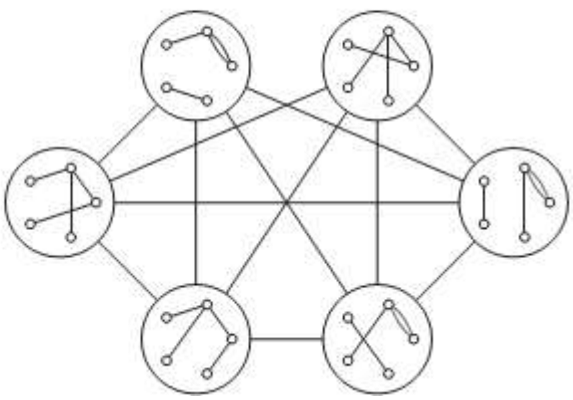


Figure 5: The reconfiguration graph R of the degree sequence (3,2,1,1,1).

$$\text{Length} = \frac{\square(G, H)}{2} - \text{mnc}\,(G, H),$$

where δ(G, H) is the size of the symmetric difference of edges and mnc(G, H) is the maximum number of alternating cycles partitioning that symmetric difference. The shortest connected graph transformation problem admits approximation algorithms; improvements and applications are discussed in the source.

III. CONCLUSION

The comprehensive overview of token-based reconfiguration problems in graph theory, highlighting their structural properties, computational complexity, and diverse applications. By examining adjacency models such as token sliding, jumping, and addition–removal, we have shown how these dynamics unify classical problems including independent set, dominating set, graph coloring, satisfiability, and degree-sequence realizations. The review consolidates known results on reachability, shortest transformation sequences, connectivity, and diameter, many of which are PSPACE-complete in general but tractable for restricted graph classes [8,9,16,19,22].

Beyond theoretical interest, reconfiguration problems connect deeply to practical domains such as robotics motion planning [4,5], physical systems [3], and algorithmic puzzles [6,7]. While significant progress has been made, several open directions remain: proving conjectures such as Cereceda's quadratic diameter bound for coloring [11,22], refining connectivity thresholds for dominating sets [20,21], and developing efficient approximation algorithms for shortest transformation sequences [23]. Addressing these challenges will not only advance discrete mathematics but also strengthen the bridge between theory and applications in computational sciences.

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