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Arrival Rate Uncertainty and the Solution Method in the Queueing Model Analysis

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Abstract- While queueing systems with random components have been thoroughly examined, this work presents an uncertain queueing model that takes the degree of customer behavior belief into account. Understanding the dynamic behavior of the underlying processes is the primary objective of mathematical modeling and analysis of queueing systems, allowing for intelligent and well-informed management choices. This study contributes to the body of research on rational consumers' decision-making in an uncertain setting where certain system parameters are random variables whose actual value is unknown to the consumers at the time of decision-making. In particular, models with a random arrival rate are of interest. Customers arrive in groups in this approach, and service and inter-arrival times are handled as unknown variables. The busy index and busy time are also explored, and the analytical formulae and uncertainty distribution are produced, respectively, in order to assess the performance of this suggested uncertain queueing model. Furthermore, several numerical examples are provided to clarify the results that were reached.

Keywords— Uncertain measures, Uncertain queueing model, and Busy period.

I. INTRODUCTION

There are numerous uses for queuing theory in a variety of industries, including computer networks, manufacturing, transportation, and inventory control. This theory explores the aggregation and dispersion phenomena and how they function in service systems. It is a subfield of mathematical operations research. The overall objective of its research is to identify statistical trends in important metrics like wait time, queue length, and busy period through mathematical modeling and performance analysis of service systems.

Queueing theory aims to obtain certain quantitative indicators to improve organizational expenses or other associated indicators by thorough statistical analyses of the inter-arrival intervals and service times of the queueing system. The queueing paradigm, which regards inter-arrival times as random variables and assumes deterministic service times, was first introduced by Erlang's groundbreaking work [1]. Initially, this model used a Poisson distribution for the customer inter-arrival time distribution. Pollaczek [22] then expanded the assumptions to include arbitrary probability distributions for service times.

As this subject developed, Pollaczek [23] investigated a multi-serve queueing model by taking into account an arbitrary probability distribution for service times and a shared Poisson distribution for interarrival periods. Lindley [4], Kendall [3], Pollaczek [24], and others continued the investigation of queueing models, which aids in comprehending the queueing phenomenon in service systems. The significance of queueing theory in resolving intricate operational problems and maximizing resource allocation across numerous areas is underscored by this multifaceted approach.

The assumptions previously made based on the conventional queueing theory framework are challenged by the introduction of various unanticipated events and emergencies in real life. The actual inter-arrival times are a sequence that is difficult to satisfy the frequency stability in the sense of probability theory, let alone the same distribution hypothesis in the sense of probability theory. In this instance, creating a distribution function for inter-arrival times based just on observational data frequently results in variations from the actual frequency of the inter-arrival times. According to queuing theory, inter-arrival times and service times are independent, identically distributed random variables that follow probability distributions.

It is important to note that researchers' empirical investigations highlight the randomness of actual observations. Examples come from a variety of domains, such as factor analysis of grain production (Liu [13]), population structure research (Liu [12]; Yang and Liu [14]), financial market analysis [12,14,17], and circuit system modeling [15]. In response to this difficulty, Liu [6] invented uncertainty theory, which was further upon by Liu [8] and gave rise to an area of mathematics that focuses on modeling degrees of conviction. Uncertainty theory has evolved over time into a broad topic of mathematics with numerous applications.

Liu [7] identified uncertain processes as an important theoretical subfield of uncertainty theory. To accurately depict the temporal evolution of uncertain phenomena, it is conceptualized as a set of uncertain variables indexed in time. The uncertain renewal process, a crucial mathematical idea of the uncertain process in which events happen constantly and independently throughout uncertain inter-arrival times, was also established at the same time. Following these foundational works, researchers both domestically and abroad have thoroughly investigated and broadened the uncertain renewal process from a variety of perspectives.

Liu [9], for example, studied the uncertain renewal reward process, a specific form that represents the cumulative rewards of a system with uncertain inter-arrival durations and uncertain awards. In order to simulate the uncertain repairable systems, Yao and Li [28] devised an uncertain alternating renewal method. Furthermore, Zhang et al. [32] investigated the uncertain delayed renewal process in order to characterize systems in which the initial inter-arrival time is substantially different from subsequent inter-arrival times. These theoretical underpinnings have allowed researchers to apply the uncertain renewal process to a variety of domains. Liu [10] took the lead in modeling insurance businesses' assets and investigating the ruin index of the uncertain insurance model in the field of insurance risk process.

Many facets of uncertain insurance models were further investigated in later research by Yao and Zhou [31], Yao and Qin [29], Liu and Yang [19, 20], and Liu [14]. The production risk process, which was first suggested by Lio and Liu [5] to address the difficulties in the production process, is another use. Block replacement strategies, which were suggested by Ke and Yao [2] as preventative replacement strategies in uncertain situations, are the subject of the third application. This strategy focuses on replacing parts in the case of a failure or at a predefined time interval. In order to ascertain the ideal replacement time for particular units, Yao and Ralescu [30] invented age replacement strategies, which constitute the fourth application.

Lastly, Yao [27] presented the fifth application, which was centered on the queueing system and represented systems that offer services in unpredictable circumstances. The uncertainty distribution of waiting time and idle time inside the uncertain queueing models was further investigated in later work by Liu and Liu [15].

However, real queueing systems are much more complex, and Yao's study [27] concentrated on the most basic types of queueing systems, such as a single-service system where clients arrive in order. Consider locations where customers prefer to arrive in groups, including restaurant windows and logistical turnaround hubs.This work aims to provide an uncertain queueing model characterized by group arrivals and determine the uncertainty distribution of the busy period in order to address the complexity of the queueing system. Furthermore, it is noteworthy that all of the results in this work appear to be less consistent with the uncertain single-service queueing model, demonstrating their applicability to a range of situations and supporting Yao's conclusions [27].

II. PRELIMINARIES

To help the reader better understand the forthcoming mathematical derivations, some of the fundamental ideas and theorems of uncertainty theory will be covered.

Definition 1: A set function defined on a universal set Γ with a σ-algebra X is called an uncertain measure μ. Three fundamental axioms are as follows: [6]

Axiom-1: (Axiom of Normality) The universal set's measure, Γ, is equal to 1.

Axiom-2: (Duality Axiom) The sum of the measure of any event E∈X and its complement Ec is 1, i.e.

$$\mu\{E\}+\mu\{E^c\}=1$$

Axiom-3: (Sub additivity Axiom) For any countable sequence of events ,..., the measure of their union is less than or equal to the sum of their individual measures, i.e.

$$\mu\left\{\bigcup_{i=1}^{\infty}E_i\right\}\le\sum_{i=1}^{\infty}\mu\{E_i\}$$

The measure of product uncertainty μ on the product σ-The following product axiom is satisfied by algebra X.

Axiom-4: (Product Axiom) If we have uncertainty spaces for i=1, 2,..., then the product uncertain measure μ satisfies

$$\mu\left\{\prod_{i=1}^{\infty}E_i\right\}=\bigwedge_{i=1}^{\infty}\mu_i\{E_i\}$$

where are arbitrarily chosen events from for i=1, 2,...,respectively.

The arithmetic of product uncertain measures is a minimum operation, whereas the arithmetic of product probability measures is a multiplication operation. Probability measures satisfy the additivity axiom, whereas uncertain measures satisfy the sub-additivity axiom.

Theorem-1:Monotonicity Theorem: The uncertain measure μ is a set function that increases monotonically.Put otherwise,

$$\mu\{E_i\}\le\mu\{E_j\}$$

for any events E_i and E_j with $E_i\subset E_j$

Liu (2007) established the notion of an uncertain variable in order to describe the values that contain uncertainty, denoted as α, which is a measurable function from an uncertainty space (Γ,X,μ) to the set of real numbers and satisfies that the set

$$\{\alpha\in B\}=\{\beta\in\Gamma:\alpha(\beta)\in B\}$$

is considered an event for any Borel set B of real numbers. Then the uncertainty distribution f(x) of α was defined by [6] as

$$f(x)=\mu\{\alpha\le x\}$$

for any real number x.

Definition – 2: Liu (2009) If an uncertainty distribution f(x) is a continuous, strictly rising function with respect to x at which 0 < f(x) <1, and approaches 0 as x tends to -∞ and 1 as x tends to ∞, it is said to be regular.

The following is a definition of a regular uncertainty distribution, which is a significant class of uncertainty distributions.

Definition –3: According to (Liu2010), Uncertain variables are considerd independent if

$$\mu\left\{\bigcap_{i=1}^{n}(\alpha\in B_i)\right\}=\bigwedge_{i=1}^{n}\mu\{\alpha\in B_i\}$$

For any real numbers $B_1,B_2,.....B_n$ Borel sets.

Some particular uncertain variables have been presented by scholars in order to model the quantities frequently used in practice. For example, Liu (2010) proposed the linear uncertain variable α~X (a, b) with a linear uncertainty distribution.

$$f(x)=\begin{cases} 0, & if\ x\le a \\ \dfrac{x-a}{b-a}, & if\ a<x\le b \\ 1, & if\ x>b \end{cases}$$

Theorem-2: Operational Law of Uncertain Variables: Assume that $\alpha_1,\alpha_2.....\alpha_n$ the independent uncertain variables have normal uncertainty distributions $f_1, f_2,....f_n$ If $g(x_1,x_2,...x_n)$ is continuous, strictly increasing with respect to $x_1,x_2,...x_m$ and strictly decreasing with respect to $x_{m+1},x_{m+2},.....x_m$ then the uncertainty distribution of $\alpha=g(\alpha_1,\alpha_2,...\alpha_n)$ is

$$f(x)=\sup_{g(x_1,x_2,....,x_n)=x}\left(\min_{1\le i\le m} f_i(x_i)\wedge\min_{m+1\le i\le n}(1-f_i(x_i))\right)$$

Theorem-3: Extreme Value Theorem: Let Xt be an independent increment process with a continuous uncertainty distribution ft(x) at each time t. Then the supremum

$$\sup_{0\le s\le t} X_s$$

has a distribution of uncertainty

$$h(x)=\inf_{0\le s\le t} f_s(x).$$

III. UNCERTAIN QUEUEING MODEL WITH GROUP ARRIVALS

An uncertain queueing model with group arrivals is first presented in this section. Examine a single server queueing system that is unpredictable and where clients arrive in large groups.

$$a_1,a_2,........$$

and with interarrival times

$$\alpha_1,\alpha_2,........$$

In particular, let the initial time of arrival be $T_0=0$ and the initial total number of customers be $A_0=0$. This means that the first group of customers arrives at time $T_1=\alpha_1$ and the total number of customers in the queueing system is $A_1=a_1$, and the second group of customers arrives at time $T_2=\alpha_1+\alpha_2$ and the total number of customers in the queueing system is, and soon.

Thus we have
$T_n=\alpha_1+\alpha_2+\cdots+\alpha_n$

and

$A_n=a_1+a_2+\cdots+a_n$

For n≥1. Service times are the amounts of time consumers spend at the counter receiving services, denoted by $\delta_1,\delta_2...$ Suppose that $a_1,a_2,..$ are iid discrete uncertain variables with positive integer values, which are called uncertain group sizes, $\alpha_1,\alpha_2,...$ are iid positive uncertain inter arrival times $\delta_1,\delta_2...$ and are iid positive uncertain service times. Then such a queueing system is called an uncertain queueing model with group arrivals.

IV. BUSY NESS AXIOM

The uncertain measure that there are always some consumers in the queueing system is known as the busy ness index, and it is one of the indicators used to assess the effectiveness of a queueing system.

Keep in mind that the system is always busy if and only if the (n+1)th group of customers consistently arrives before the nth group of customers departs, for any positive integer n.Specifically, we have

$$\sum_{i=1}^{n+1}\alpha_i-\left(\alpha_1+\sum_{j=1}^{A_n}\delta_j\right)=\sum_{i=2}^{n+1}\alpha_i-\sum_{j=1}^{A_n}\delta_j\le 0$$

For $n = 1, 2, \ldots\ldots$ we write

$$\Upsilon_n = \sum_{i=2}^{n+1} \alpha_i - \sum_{j=1}^{A_n} \delta_j = b_{h^*} \vee a_{h^*}.$$

Consequently $n = 1, 2, \ldots\ldots$, the busyness index may be represented by

Busyness $= \mu\left\{\max_{n\ge 1} \Upsilon_n \le 0\right\}.$

In order to derive the activity index expression, we first present a helpful lemma.

Lemma – 1: Let $\langle a_n \rangle$ be a decreasing sequence and $\langle b_n \rangle$ be an increasing sequence. If

$$\lim_{n\to\infty} a_n < \lim_{n\to\infty} b_n,$$

then

$$\min_{n\ge 1} a_n \vee b_n = \max_{n\ge 1} a_{n-1} \wedge b_n \qquad [4.1]$$

where a_0 is a number satisfying $a_0 \ge a_1 \vee b_1.$

Proof: If $a_1 \le b_1,$ then we have

$$a_n \le a_1 \le b_1 \le b_n \le b_{n+1}$$

for any $n \ge 1$. Therefore, we can get

$$\min_{n\ge 1} a_n \vee b_n = \min_{n\ge 1} b_n = b_1$$

and

$$\begin{aligned}\max_{n\ge 1} a_{n-1} \wedge b_n &= (a_0 \wedge b_1) \vee \max_{n\ge 1} a_n \wedge b_{n+1} \\ &= b_1 \vee \max_{n\ge 1} a_n \\ &= b_1.\end{aligned}$$

If $a_1 > b_1$, then we define a positive integer as follows,

$$h^* = \inf_{h\ge 1}\left\{h \mid a_h \vee b_h = \min_{n\ge 1} a_n \vee b_n\right\},$$

and the argument breaks down into two cases.

Case – 1: When $b_{h^*} \le a_{h^*},$ we have

$$a_{h^*} = a_{h^*} \vee b_{h^*} \le a_{h^*+1} \vee b_{h^*+1}.$$

It follows from $a_{h^*+1} \le a_{h^*}$ that

$a_{h^*} \le b_{h^*+1}.$

Then we have

$b_n \le b_{h^*} \le a_{h^*} \le a_{n-1}.$

for any $n \le h^*$ and

$$a_{n-1} \le a_{h^*} \le b_{h^*+1} \le b_n$$

for any $n \ge h^* + 1$. Therefore, we can get

$$\begin{aligned}\max_{n\ge 1} a_{n-1} \wedge b_n &= \left(\max_{n\le h^*} a_{n-1} \wedge b_n\right) \vee \left(\max_{n\ge h^*+1} a_{n-1} \wedge b_n\right) \\ &= \left(\max_{n\le h^*} b_n\right) \vee \left(\max_{n\ge h^*+1} a_{n-1}\right) \\ &= b_{h^*} \vee a_{h^*}.\end{aligned}$$

Case – 2: When $a_{h^*} < b_{h^*},$ we have

$$b_{h^*} = a_{h^*} \vee b_{h^*} < a_{h^*-1} \vee b_{h^*-1}.$$

It follows from $b_{h^*-1} \le b_{h^*}$ that

$$b_{h^*} \le a_{h^*-1}.$$

Then we have

$$b_n \le b_{h^*} < a_{h^*-1} \le a_{n-1}$$

for any $n \le h^*$ and

$$a_{n-1} \le a_{h^*} < b_{h^*} \le b_n$$

for any $n \ge h^* + 1$. Therefore, we can get

$$\max_{n\ge 1} a_{n-1}\wedge b_n=\left(\max_{n\le h^*} a_{n-1}\wedge b_n\right)\vee\left(\max_{n\ge h^*+1} a_{n-1}\wedge b_n\right)$$

$$=\left(\max_{n\le h^*} b_n\right)\vee\left(\max_{n\ge h^*+1} a_{n-1}\right)$$

$$=b_{h^*}\vee a_{h^*}.$$

Hence, the lemma is proved.

Theorem – 4: Examine an unsure group in an uncertain queueing system,size $a_1,a_2,..$ iid positive uncertain interarrival time $\alpha_1,\alpha_2,...$ are iid positive uncertain service times $\delta_1,\delta_2....$ Assume $(a_1,a_2),(\alpha_1,\alpha_2)$ and (δ_1,δ_2) are independent uncertain vectors, Interarrival times and service times have regular uncertainty distributions $\mathcal{F}$ and G, respectively, but those group sizes have a shared uncertainty distribution $\mathcal{H}$.Then

$$Busyness=\min_{r\ge 1}\max_{s\ge r}\sup_{x\ge 0}\left(1-\mathrm{H}\left(\frac{s-1}{r}\right)\right)\wedge \mathrm{F}\left(\frac{x}{r}\right)\wedge\left(1-G\left(\frac{x}{s}\right)\right). \quad [4.2]$$

Proof: we know that

Busyness $=\mu\left\{\max_{n\ge 1}\Upsilon_n\le 0\right\}$

where

$$\Upsilon_n=\sum_{i=2}^{n+1}\alpha_i-\sum_{j=1}^{A_n}\delta_j\ ,$$

First, we determine the uncertainty distributions of ($\Upsilon_n\ n=1,2......$). Using Theorem 1, independent and operational lows of uncertain variables, and Lemma 1, for every non-negative positive integer r and any given y ∈ ℝ , we obtain

$$\mu\{\Upsilon_r\le y\}=\mu\left\{\sum_{i=2}^{r+1}\alpha_i-\sum_{j=1}^{A_r}\delta_j\le y\right\}$$

$$=\mu\left\{\bigcup_{s=r}^{\infty}\left(A_r=s,\sum_{i=2}^{r+1}\alpha_i-\sum_{j=1}^{s}\delta_j\le y\right)\right\}$$

$$=\mu\left\{\bigcup_{s=r}^{\infty}\left(A_r\ge s,\sum_{i=2}^{r+1}\alpha_i-\sum_{j=1}^{s}\delta_j\le y\right)\right\}$$

$$=\mu\left\{\bigcup_{s=r}^{\infty}\left(\sum_{i=1}^{r}a_i>s-1,\sum_{i=2}^{r+1}\alpha_i-\sum_{j=1}^{s}\delta_j\le y\right)\right\}$$

$$\ge\max_{s\ge r}\mu\left\{\sum_{i=1}^{r}a_i>s-1,\sum_{i=2}^{r+1}\alpha_i-\sum_{j=1}^{s}\delta_j\le y\right\}$$

$$=\max_{s\ge r}\mu\left\{\sum_{i=1}^{r}a_i>s-1\right\}\wedge\mu\left\{\sum_{i=2}^{r+1}\alpha_i-\sum_{j=1}^{s}\delta_j\le y\right\}$$

$$=\max_{s\ge r}\left(1-\mathrm{H}\left(\frac{s}{r}\right)\right)\wedge\left(\sup_{x\ge 0}\mathrm{F}\left(\frac{x}{r}\right)\wedge\left(1-G\left(\frac{x-y}{s}\right)\right)\right)$$

$$=\min_{s\ge r}\left(1-\mathrm{H}\left(\frac{s}{r}\right)\right)\vee\left(\inf_{x\ge 0}\mathrm{F}\left(\frac{x}{r}\right)\vee\left(1-G\left(\frac{x-y}{s}\right)\right)\right)$$

$$=\min_{s\ge r}\inf_{x\ge 0}\left(1-\mathrm{H}\left(\frac{s}{r}\right)\right)\vee\mathrm{F}\left(\frac{x}{r}\right)\vee\left(1-G\left(\frac{x-y}{s}\right)\right)$$

However, by applying Theorem 1, operational lows of unknown variables, and independence, we may obtain

$$\mu\{\Upsilon_r\le y\}=1-\mu\{\Upsilon_r>y\}$$

$$=1-\mu\left\{\sum_{i=2}^{r+1}\alpha_i-\sum_{j=1}^{A_r}\delta_j>y\right\}$$

$$=1-\mu\left\{\bigcup_{s=r}^{\infty}\left(A_r=s,\sum_{i=2}^{r+1}\alpha_i-\sum_{j=1}^{s}\delta_j>y\right)\right\}$$

$$=1-\mu\left\{\bigcup_{s=r}^{\infty}\left(A_r\le s,\sum_{i=2}^{r+1}\alpha_i-\sum_{j=1}^{s}\delta_j>y\right)\right\}$$

$$=1-\mu\left\{\bigcup_{s=r}^{\infty}\left(\sum_{i=1}^{r}a_i \le s, \sum_{i=2}^{r+1}\alpha_i-\sum_{j=1}^{s}\delta_j > y\right)\right\}$$

$$\le 1-\max_{s\ge r}\mu\left\{\sum_{i=1}^{r}a_i \le s, \sum_{i=2}^{r+1}\alpha_i-\sum_{j=1}^{s}\delta_j > y\right\}$$

$$=1-\max_{s\ge r}\mu\left\{\sum_{i=1}^{r}a_i \le s\right\}\wedge\mu\left\{\sum_{i=2}^{r+1}\alpha_i-\sum_{j=1}^{s}\delta_j > y\right\}$$

$$=1-\max_{s\ge r}\mathrm{H}\left(\frac{s}{r}\right)\wedge\left(1-\sup_{x\ge 0}\mathrm{F}\left(\frac{x}{r}\right)\wedge\left(1-G\left(\frac{x-y}{s}\right)\right)\right)$$

$$=\min_{s\ge r}\left(1-\mathrm{H}\left(\frac{s}{r}\right)\right)\vee\left(\inf_{x\ge 0}\mathrm{F}\left(\frac{x}{r}\right)\vee\left(1-G\left(\frac{x-y}{s}\right)\right)\right)$$

$$=\min_{s\ge r}\inf_{x\ge 0}\left(1-\mathrm{H}\left(\frac{s}{r}\right)\right)\vee\mathrm{F}\left(\frac{x}{r}\right)\vee\left(1-G\left(\frac{x-y}{s}\right)\right)$$

the distribution of uncertainty for is

$$\mathrm{N}_r(y)=\min_{s\ge r}\inf_{x\ge 0}\left(1-\mathrm{H}\left(\frac{s}{r}\right)\right)\vee\mathrm{F}\left(\frac{x}{r}\right)\vee\left(1-G\left(\frac{x-y}{s}\right)\right)$$

.

Theorem 3 indicates that has an uncertainty distribution since Υ_r it is an independent increment process that has an uncertainty distribution since it is an independent increment process.

$$\mathrm{P}(y)=\min_{r\ge 1}\mathrm{N}_r(y)$$

Then we have

Busyness $=\mu\left\{\max_{r\ge 1}\Upsilon_r \le 0\right\}=\min_{r\ge 1}\mathrm{N}_r(0)$

Remark-1: The uncertain queueing model with group arrivals becomes the uncertain single service queueing model that Yao examined when the uncertain group sizes a1,a2 .....always constant 1,the uncertain queueing model with group arrivals becomes uncertain single service queueing model investigated by Yao. In this case, we have H

$$\mathrm{H}(x)=\begin{cases}0, & if\ x<1\\ 1, & if\ x\ge 1\end{cases}$$

Hence we can conclude that the activity index's analytical expression is

Busyness

$$=\min_{r\ge 1}\min_{s\ge r}\inf_{x\ge 0}\left(1-\mathrm{H}\left(\frac{s}{r}\right)\right)\vee\mathrm{F}\left(\frac{x}{r}\right)\vee\left(1-G\left(\frac{x-y}{s}\right)\right)$$

$$=\min_{r\ge 1}\min_{s\ge r}\inf_{x\ge 0}\mathrm{F}\left(\frac{x}{r}\right)\vee\left(1-G\left(\frac{x}{s}\right)\right)$$

$$=\min_{r\ge 1}\inf_{x\ge 0}\mathrm{F}\left(\frac{x}{r}\right)\vee\left(1-G\left(\frac{x}{r}\right)\right)$$

$$=\inf_{x\ge 0}\mathrm{F}(x)\vee\left(1-G(x)\right)$$

$$=\sup_{x\ge 0}\mathrm{F}(x)\wedge\left(1-G(x)\right)$$

The result of Yao's uncertain single-service queueing model simplifies this.

V. BUSY PERIOD

When there are constantly some clients in the queueing system, it is referred to as a busy period. For any time when things are busy, since $a_1, a_2,..$ are iid uncertain group sizes, $\alpha_1,\alpha_2,...$ .are iid uncertain inter arrival times $\delta_1,\delta_2...$ are iid uncertain service times, Without losing generality, we can presume that the first group of clients arrives at the beginning of the busy period. In this step, we can also reset the busy period's beginning time to 0. Let's take a moment before the next busy period and after the present one. Then B_{t+1} groups (including the first group) of customers arriving before the time t and the total number of customers is

$$A_t = \sum_{i=1}^{N_t+1} a_i,$$

wherer there is an unpredictable inter-arrival time $\alpha_1, \alpha_2, \ldots$. renewal procedure. The total service time for those customers is less than t because the busy period finishes before t.

Consequently, the busy period is the lowest value of $\sum_{j=1}^{A_t} \delta_j < t$ such that

That is, the busy period might be represented by

$$\tau = \inf\left\{ t > 0 : \sum_{j=1}^{A_t} \delta_j < t \right\}$$

Theorem-5 Take into consideration an uncertain queueing system with iid unknown group sizes $a_1, a_2, \ldots$ iid positive uncertain interarrival times $\alpha_1, \alpha_2, \ldots$ are iid Positive unpredictable service periods $\delta_1, \delta_2 \ldots$. Assume $(a_1, a_2), (\alpha_1, \alpha_2)$ and (δ_1, δ_2) are separate uncertain vectors that share a common uncertainty distribution ($\mathcal{H}$), while interarrival times and service times have regular uncertainty distributions ($\mathcal{F}$ and G, respectively). The distribution of τ's uncertainty is thus

$$Q(t) = \max_{r \ge 1} \max_{s \ge r} \sup_{x \le t} \mathrm{H}\left(\frac{s}{r}\right) \wedge \left(1 - \mathrm{F}\left(\frac{x}{s}\right)\right) \wedge G\left(\frac{x}{s}\right). \quad (5.1)$$

Proof: For any given time t, the busy period if and only if, for some positive integer r, the rth group of customers depart before the arrival of the (r+1)st group of customers and before the time t. That is,

$$\{\tau \le t\} = \bigcup_{r=1}^{\infty} \left\{ \sum_{j=1}^{A_r} \delta_j < \sum_{i=2}^{r+1} \alpha_i, \sum_{j=1}^{A_r} \delta_j \le t \right\}$$

$$= \bigcup_{r=1}^{\infty} \bigcup_{s=r}^{\infty} \left\{ A_r = s, \sum_{j=1}^{s} \delta_j < \sum_{i=2}^{r+1} \alpha_i, \sum_{j=1}^{s} \delta_j \le t \right\} \quad (5.2)$$

$$= \bigcup_{r=1}^{\infty} \bigcup_{s=r}^{\infty} \left\{ A_r \le s, \sum_{j=1}^{s} \delta_j < \sum_{i=2}^{r+1} \alpha_i, \sum_{j=1}^{s} \delta_j \le t \right\}$$

$$= \bigcup_{r=1}^{\infty} \bigcup_{s=r}^{\infty} \left\{ \sum_{i=1}^{r} a_i \le s, \sum_{j=1}^{s} \delta_j < \sum_{i=2}^{r+1} \alpha_i, \sum_{j=1}^{s} \delta_j \le t \right\}$$

.

Next, given positive integers r and s (s ≥ r), A number is defined as

$$\beta_{rs} = \sup_{x \le t} \mathrm{H}\left(\frac{s}{r}\right) \wedge \left(1 - \mathrm{F}\left(\frac{x}{r}\right)\right) \wedge G\left(\frac{x}{s}\right). \quad (5.3)$$

and prove that

$$Q_t = \max_{r \ge 1} \max_{s \ge r} \beta_{rs} \quad (5.4)$$

The usual uncertainty distribution $\mathcal{H}$ of group sizes is irregular because the uncertain group size always takes positive integer values. The inverse function of $\mathcal{H}$ is therefore defined by

$$\mathrm{H}^{-1}(\beta) = \max_{x \in \mathbb{N}} \{ x : \mathrm{H}(x) \le \beta \}$$

for anyone It is easy to infer that

$$\mathrm{H}^{-1}(\mathrm{H}(x)) = \mathrm{H}^{-1}\left(\mathrm{H}\left(\lfloor x \rfloor\right)\right) = \lfloor x \rfloor \quad (5.5)$$

for anyone $x > 0$ and

$$\mathrm{H}\left(\mathrm{H}^{-1}(\beta)\right) \le \beta \quad (5.6)$$

for anyone $\beta \in (0,1).$ For convenience, we have

$$g^{-1}(0) = \lim_{\beta \to 0^+} g^{-1}(\beta), g^{-1}(1) = \lim_{\beta \to 1^-} g^{-1}(\beta)$$

for every distribution of uncertainty g.
The argument then divides into three scenarios.

Case – 1: Suppose

$$\beta_{rs} = 0, r = 1, 2, \ldots\ldots, s = r, r+1, \ldots\ldots$$

Given any two positive numbers, r and $(s \ge r)$, (5.3) leads to the conclusion that

$$\mathrm{H}\left(\frac{s}{r}\right)=0 \quad (5.7)$$

or

$$\sup_{x\le t}\left(1-\mathrm{F}\left(\frac{x}{r}\right)\right)\wedge G\left(\frac{x}{s}\right)=0. \quad (5.8)$$

When eq. (5.8) holds, if $sG^{-1}(0)<t$, then we have

$$G\left(\frac{t}{s}\right)>0.$$

Since the $\sup_{x\le t}\left(1-\mathrm{F}\left(\frac{x}{r}\right)\right)\wedge G\left(\frac{x}{s}\right)=0$, we can infer that

$$1-\mathrm{F}\left(\frac{x}{r}\right)=0$$

at the range of $sG^{-1}(0)<x\le t$ ($sG^{-1}(0)$ is the largest point that makes $G(t/s)=0$ holds). That is,

$$\mathrm{F}\left(\frac{x}{r}\right)=1$$

Considering the continuity and montonicity of F, at the range of $sG^{-1}(0)<x\le t$, we have

$$\mathrm{F}\left(\frac{sG^{-1}(0)}{r}\right)=1$$

Hence we have

$$\frac{sG^{-1}(0)}{r}\ge \mathrm{F}^{-1}(1).$$

That is,

$$r\mathrm{F}^{-1}(1)\le sG^{-1}(0)$$

and after that, we get

$$sG^{-1}(0)<t, r\mathrm{F}^{-1}(1)\le sG^{-1}(0) \quad (5.9)$$

If not, we have

$$sG^{-1}(0)\ge t. \quad (5.10)$$

Consequently, one of the options (5.7)–(5.10) is true. Using (5.7)–(5.10), we obtain

$$\mu\left\{\sum_{i=1}^{r}a_i\le s,\sum_{j=1}^{s}\delta_j<\sum_{i=2}^{r+1}\alpha_i,\sum_{j=1}^{s}\delta_j\le t\right\}$$

$$\le\mu\left\{\left(\bigcup_{i=1}^{r}\left(a_i\le\frac{s}{r}\right)\right)\cup\left(\bigcup_{j=1}^{s}\left(\delta_j\le G^{-1}(0)\right)\right)\cup\left(\bigcup_{i=2}^{r+1}\left(\alpha_i>\mathrm{F}^{-1}(1)\right)\right)\right\}$$

$$=\mu\left\{\bigcup_{i=1}^{r}\left(a_i\le\frac{s}{r}\right)\right\}\vee\mu\left\{\bigcup_{j=1}^{s}\left(\delta_j\le G^{-1}(0)\right)\right\}\vee\mu\left\{\bigcup_{i=2}^{r+1}\left(\alpha_i>\mathrm{F}^{-1}(1)\right)\right\}$$

$$=\bigvee_{i=1}^{r}\mu\left\{a_i\le\frac{s}{r}\right\}\vee\bigvee_{j=1}^{s}\mu\left\{\delta_j\le G^{-1}(0)\right\}\vee\bigvee_{i=2}^{r+1}\mu\left\{\alpha_i>\mathrm{F}^{-1}(1)\right\}=0$$

That is,

$$\mu\left\{\sum_{i=1}^{r}a_i\le s,\sum_{j=1}^{s}\delta_j<\sum_{i=2}^{r+1}\alpha_i,\sum_{j=1}^{s}\delta_j\le t\right\}=0 \quad (5.11)$$

when $\beta_{rs}=0.$ Therefore, it follows from (5.2), (5.11) and sub additivity axiom [6] that

$$\begin{aligned}Q(t)&=\mu\{\tau\le t\}\\&=\mu\left\{\bigcup_{r=1}^{\infty}\bigcup_{s=r}^{\infty}\left(\sum_{i=1}^{r}a_i\le s,\sum_{j=1}^{s}\delta_j<\sum_{i=2}^{r+1}\alpha_i,\sum_{j=1}^{s}\delta_j\le t\right)\right\}\\&\le\sum_{r\ge1,s\ge r}\mu\left\{\sum_{i=1}^{r}a_i\le s,\sum_{j=1}^{s}\delta_j<\sum_{i=2}^{r+1}\alpha_i,\sum_{j=1}^{s}\delta_j\le t\right\}\\&=0\end{aligned}$$

After then (5.4) holds.

Case - 2: Assume that some integers exist $r^*\ge1 \, and \, s^*\ge r^*$ such that $\beta_{r^*s^*}=1$

It follows from (5.3) that we have

$$\mathrm{H}\left(\frac{s^*}{r^*}\right)=\sup_{x\le t}\left(1-\mathrm{F}\left(\frac{x}{s^*}\right)\right)\wedge G\left(\frac{x}{s^*}\right)=1.$$

Thus

$$\mathrm{H}\left(\frac{s^*}{r^*}\right)=1, G\left(\frac{t}{s^*}\right)\ge 1, G\left(\frac{r^*\mathrm{F}^{-1}(0)}{s^*}\right)\ge 1.$$

That is,

$$\mathrm{H}\left(\frac{s^*}{r^*}\right)=1, s^*G^{-1}(1)\le t,$$

$$s^*G^{-1}(1)\le r^*\mathrm{F}^{-1}(0). \quad (5.12)$$

By using (5.12) and sub additivity axiom [6], we can get

$$Q(t)=\mu\{\tau\le t\}$$

$$=\mu\left\{\bigcup_{r=1}^{\infty}\bigcup_{s=r}^{\infty}\left(\sum_{i=1}^{r}a_i\le s, \sum_{j=1}^{s}\delta_j<\sum_{i=2}^{r+1}\alpha_i, \sum_{j=1}^{s}\delta_j\le t\right)\right\}$$

$$\ge\mu\left\{\sum_{i=1}^{r^*}a_i\le s^*, \sum_{j=1}^{s^*}\delta_j<\sum_{i=2}^{r^*+1}\alpha_i, \sum_{j=1}^{s^*}\delta_j\le t\right\}$$

$$\ge\mu\left\{\left(\bigcap_{i=1}^{r^*}\left(a_i\le\frac{s^*}{r^*}\right)\right)\cap\left(\bigcap_{j=1}^{s^*}\left(\delta_j\le G^{-1}(1)\right)\right)\cap\left(\bigcap_{i=2}^{r^*+1}\left(\alpha_i>\mathrm{F}^{-1}(0)\right)\right)\right\}$$

$$=\mu\left\{\bigcap_{i=1}^{r^*}\left(a_i\le\frac{s^*}{r^*}\right)\right\}\wedge\mu\left\{\bigcap_{j=1}^{s^*}\left(\delta_j\le G^{-1}(1)\right)\right\}\wedge\mu\left\{\bigcap_{i=2}^{r^*+1}\left(\alpha_i>\mathrm{F}^{-1}(0)\right)\right\}$$

$$=\bigwedge_{i=1}^{r^*}\mu\left\{a_i\le\frac{s^*}{r^*}\right\}\wedge\bigwedge_{j=1}^{s^*}\mu\left\{\delta_j\le G^{-1}(1)\right\}\wedge\bigwedge_{i=2}^{r^*+1}\mu\left\{\alpha_i>\mathrm{F}^{-1}(0)\right\}=1$$

after which (5.4) holds

Case-3: Suppose

$$\beta_{rs}<1, r=1,2,\ldots\ldots, s=r,r+1,\ldots\ldots$$

and a nonempty subset E of the index set exists such that

$$\beta_{rs}>0$$

for anyone $(r,s)\in E$, for any $(r,s)\in E$ we than have

$$0<\beta_{rs}<1.$$

Given that definition 2 defines G as a normal uncertainty distribution, G is a continuous function that strictly increases with respect to x at which $0<G(x)<1$, and then $G\left(\frac{x}{s}\right)$ is additionally a strictly growing, continuous function with respect to x at which $0<G\left(\frac{x}{s}\right)<1.$ Then it follows from $0<\beta_{rs}<1$ that there needs to be a unique x^* satisfying that

$$G\left(\frac{x^*}{s}\right)=\beta_{rs}.$$

If $\beta_{rs}=\mathrm{H}\left(\frac{s}{r}\right)\le\sup_{x\le t}\left(1-\mathrm{F}\left(\frac{x}{r}\right)\right)\wedge G\left(\frac{x}{s}\right),$ then we have

$$\beta_{rs}=\mathrm{H}\left(\frac{s}{r}\right), \beta_{rs}=G\left(\frac{x^*}{s}\right), \beta_{rs}\le 1-\mathrm{F}\left(\frac{x^*}{r}\right), x^*\le t.$$

That is,

$$\mathrm{H}^{-1}(\beta_{rs})=\left\lfloor\frac{s}{r}\right\rfloor, G^{-1}(\beta_{rs})=\frac{x^*}{s} \text{ and}$$

$$\mathrm{F}^{-1}(1-\beta_{rs})\ge\frac{x^*}{r}, x^*\le t.$$

Thus,

$$r\mathrm{H}^{-1}(\beta_{rs})>s-r, sG^{-1}(\beta_{rs})\le r\mathrm{F}^{-1}(1-\beta_{rs}), sG^{-1}(\beta_{rs})\le t. \quad (5.13)$$

If

$$\beta_{rs}=\sup_{x\le t}\left(1-\mathrm{F}\left(\frac{x}{r}\right)\right)\wedge G\left(\frac{x}{s}\right)<\mathrm{H}\left(\frac{s}{r}\right).$$

and $x^* = t$, then we have

$$\beta_{rs} < \mathrm{H}\left(\frac{s}{r}\right), \beta_{rs} = G\left(\frac{t}{s}\right), \beta_{rs} \le 1 - \mathrm{F}\left(\frac{t}{r}\right).$$

That is,

$$\mathrm{H}^{-1}(\beta_{rs}) = \left\lfloor \frac{s}{r} \right\rfloor, G^{-1}(\beta_{rs}) = \frac{t}{s},$$

$$\mathrm{F}^{-1}(1-\beta_{rs}) \ge \frac{t}{r}.$$

Thus,

$$r\mathrm{H}^{-1}(\beta_{rs}) \le s,\ sG^{-1}(\beta_{rs}) \le r\mathrm{F}^{-1}(1-\beta_{rs}),\ sG^{-1}(\beta_{rs}) = t. \quad (5.14)$$

If

$$\beta_{rs} = \sup_{x \le t}\left(1 - \mathrm{F}\left(\frac{x}{r}\right)\right) \wedge G\left(\frac{x}{s}\right) < \mathrm{H}\left(\frac{s}{r}\right)$$

and $x^* < t,$ then we have

$$\beta_{rs} < \mathrm{H}\left(\frac{s}{r}\right), \beta_{rs} = G\left(\frac{x^*}{s}\right), \beta_{rs} = 1 - \mathrm{F}\left(\frac{x^*}{r}\right), x^* < t.$$

That is,

$$\mathrm{H}^{-1}(\beta_{rs}) \le \left\lfloor \frac{s}{r} \right\rfloor, G^{-1}(\beta_{rs}) = \frac{x^*}{s},$$

$$\mathrm{F}^{-1}(1-\beta_{rs}) = \frac{x^*}{r},\ x^* < t.$$

Thus,

$$r\mathrm{H}^{-1}(\beta_{rs}) \le s,\ sG^{-1}(\beta_{rs}) \le r\mathrm{F}^{-1}(1-\beta_{rs}),\ sG^{-1}(\beta_{rs}) \le t. \quad (5.15)$$

Therefore, one of the alternatives (5.13)-(5.15) holds. Let us turn our discussion to the uncertainty distribution of τ. On the other hand, by using (5.13)-(5.15), we obtain

$$\begin{aligned}
Q(t) &= \mu\{\tau \le t\} \\
&= \mu\left\{\bigcup_{r=1}^{\infty}\bigcup_{s=r}^{\infty}\left(\sum_{i=1}^{r} a_i \le s, \sum_{j=1}^{s}\delta_j < \sum_{i=2}^{r+1}\alpha_i, \sum_{j=1}^{s}\delta_j \le t\right)\right\} \\
&\ge \max_{(r,s)\in E}\mu\left\{\sum_{i=1}^{r} a_i \le s, \sum_{j=1}^{s}\delta_j < \sum_{i=2}^{r+1}\alpha_i, \sum_{j=1}^{s}\delta_j \le t\right\} \\
&\ge \max_{(r,s)\in E}\mu\left\{\left(\bigcap_{i=1}^{r}\left(a_i \le \frac{s}{r}\right)\right) \cap \left(\bigcap_{j=1}^{s}\left(\delta_j \le G^{-1}(\beta_{rs})\right)\right) \cap \left(\bigcap_{i=2}^{r+1}\left(\alpha_i > \mathrm{F}^{-1}(1-\beta_{rs})\right)\right)\right\} \\
&= \max_{(r,s)\in E}\mu\left\{\bigcap_{i=1}^{r}\left(a_i \le \frac{s}{r}\right)\right\} \wedge \mu\left\{\bigcap_{j=1}^{s}\left(\delta_j \le G^{-1}(\beta_{rs})\right)\right\} \wedge \mu\left\{\bigcap_{i=2}^{r+1}\left(\alpha_i > \mathrm{F}^{-1}(1-\beta_{rs})\right)\right\} \\
&= \max_{(r,s)\in E}\bigwedge_{i=1}^{r}\mu\left\{a_i \le \frac{s}{r}\right\} \wedge \bigwedge_{j=1}^{s}\mu\left\{\delta_j \le G^{-1}(\beta_{rs})\right\} \wedge \bigwedge_{i=2}^{r+1}\mu\left\{\alpha_i > \mathrm{F}^{-1}(1-\beta_{rs})\right\} \\
&= \max_{(r,s)\in E}\bigwedge_{i=1}^{r}\mathrm{H}\left(\frac{s}{r}\right) \wedge \bigwedge_{j=1}^{s}\beta_{rs} \wedge \bigwedge_{i=2}^{r+1}\beta_{rs} \\
&= \max_{(r,s)\in E}\bigwedge_{i=1}^{r}\mathrm{H}\left(\frac{s}{r}\right) \wedge \beta_{rs} \\
&= \max_{(r,s)\in E}\beta_{rs} \\
&= \max_{r\ge 1}\max_{s\ge r}\beta_{rs}
\end{aligned}$$

However, it is clear from the definition of E that

$$\beta_{rs} = 0$$

for any $(r,s)\in E^c$. Then using equation (5.11), we have

$$\sum_{(r,s)\in E^c}\mu\left\{\sum_{i=1}^{r} a_i \le s, \sum_{j=1}^{s}\delta_j < \sum_{i=2}^{r+1}\alpha_i, \sum_{j=1}^{s}\delta_j \le t\right\} = 0 \quad (5.16)$$

By using (5.13)-(5.16) and subadditivity axiom [6], we have

$$Q(t)=\mu\{\tau\le t\}$$

$$=\mu\left\{\bigcup_{r=1}^{\infty}\bigcup_{s=r}^{\infty}\left(\sum_{i=1}^{r}a_i\le s,\sum_{j=1}^{s}\delta_j<\sum_{i=2}^{r+1}\alpha_i,\sum_{j=1}^{s}\delta_j\le t\right)\right\}$$

$$\le\sum_{(r,s)\in E^c}\mu\left\{\sum_{i=1}^{r}a_i\le s,\sum_{j=1}^{s}\delta_j<\sum_{i=2}^{r+1}\alpha_i,\sum_{j=1}^{s}\delta_j\le t\right\}+\mu\left\{\bigcup_{(r,s)\in E}\left(\sum_{i=1}^{r}a_i\le s,\sum_{j=1}^{s}\delta_j<\sum_{i=2}^{r+1}\alpha_i,\sum_{j=1}^{s}\delta_j\le t\right)\right\}$$

$$\le\mu\left\{\bigcup_{(r,s)\in E}\left(\sum_{i=1}^{r}a_i\le s,\sum_{j=1}^{s}\delta_j<\sum_{i=2}^{r+1}\alpha_i,\sum_{j=1}^{s}\delta_j\le t\right)\right\}$$

$$\le\mu\left\{\bigcup_{(r,s)\in E}\left(\left(\bigcup_{i=1}^{r}\left(a_i\le \mathrm{H}^{-1}(\beta_{rs})\right)\right)\cup\left(\bigcup_{j=1}^{s}\left(\delta_j\le G^{-1}(\beta_{rs})\right)\right)\cup\left(\bigcup_{i=2}^{r+1}\left(\alpha_i>\mathrm{F}^{-1}(1-\beta_{rs})\right)\right)\right)\right\}$$

$$\le\mu\left\{\bigcup_{r=1}^{\infty}\bigcup_{s=r}^{\infty}\left(\left(\bigcup_{i=1}^{r}\left(a_i\le \mathrm{H}^{-1}(\beta_{rs})\right)\right)\cup\left(\bigcup_{j=1}^{s}\left(\delta_j\le G^{-1}(\beta_{rs})\right)\right)\cup\left(\bigcup_{i=2}^{r+1}\left(\alpha_i>\mathrm{F}^{-1}(1-\beta_{rs})\right)\right)\right)\right\}$$

$$=\mu\left\{\left(\bigcup_{r=1}^{\infty}\bigcup_{s=r}^{\infty}\bigcup_{i=1}^{r}\left(a_i\le \mathrm{H}^{-1}(\beta_{rs})\right)\right)\cup\left(\bigcup_{r=1}^{\infty}\bigcup_{s=r}^{\infty}\bigcup_{j=1}^{s}\left(\delta_j\le G^{-1}(\beta_{rs})\right)\right)\cup\left(\bigcup_{r=1}^{\infty}\bigcup_{s=r}^{\infty}\bigcup_{i=2}^{r+1}\left(\alpha_i>\mathrm{F}^{-1}(1-\beta_{rs})\right)\right)\right\}$$

$$=\mu\left\{\bigcup_{r=1}^{\infty}\bigcup_{s=r}^{\infty}\bigcup_{i=1}^{r}\left(a_i\le \mathrm{H}^{-1}(\beta_{rs})\right)\right\}\vee\mu\left\{\bigcup_{r=1}^{\infty}\bigcup_{s=r}^{\infty}\bigcup_{j=1}^{s}\left(\delta_j\le G^{-1}(\beta_{rs})\right)\right\}\vee\mu\left\{\bigcup_{r=1}^{\infty}\bigcup_{s=r}^{\infty}\bigcup_{i=2}^{r+1}\left(\alpha_i>\mathrm{F}^{-1}(1-\beta_{rs})\right)\right\}$$

$$=\mu\left\{\bigcup_{r=1}^{\infty}\bigcup_{i=1}^{r}\bigcup_{s=r}^{\infty}\left(a_i\le \mathrm{H}^{-1}(\beta_{rs})\right)\right\}\vee\mu\left\{\bigcup_{r=1}^{\infty}\bigcup_{j=1}^{\infty}\bigcup_{s=r\vee j}^{\infty}\left(\delta_j\le G^{-1}(\beta_{rs})\right)\right\}\vee\mu\left\{\bigcup_{r=1}^{\infty}\bigcup_{i=2}^{r+1}\bigcup_{s=r}^{\infty}\left(\alpha_i>\mathrm{F}^{-1}(1-\beta_{rs})\right)\right\}$$

$$=\mu\left\{\bigcup_{i=1}^{\infty}\bigcup_{r=i}^{\infty}\bigcup_{s=r}^{\infty}\left(a_i\le \mathrm{H}^{-1}(\beta_{rs})\right)\right\}\vee\mu\left\{\bigcup_{j=1}^{\infty}\bigcup_{r=1}^{\infty}\bigcup_{s=r\vee j}^{\infty}\left(\delta_j\le G^{-1}(\beta_{rs})\right)\right\}\vee\mu\left\{\bigcup_{i=2}^{\infty}\bigcup_{r=i-1}^{\infty}\bigcup_{s=r}^{\infty}\left(\alpha_i>\mathrm{F}^{-1}(1-\beta_{rs})\right)\right\}$$

$$=\mu\left\{\bigcup_{i=1}^{\infty}\left(a_i\le\max_{r\ge i}\max_{s\ge r}\mathrm{H}^{-1}(\beta_{rs})\right)\right\}\vee\mu\left\{\bigcup_{j=1}^{\infty}\left(\delta_j\le\max_{r\ge 1}\max_{s\ge r\vee j}G^{-1}(\beta_{rs})\right)\right\}\vee\mu\left\{\bigcup_{i=2}^{\infty}\left(\alpha_i>\min_{r\ge i-1}\min_{s\ge r}\mathrm{F}^{-1}(1-\beta_{rs})\right)\right\}$$

.

$$\le\mu\left\{\bigcup_{i=1}^{\infty}\left(a_i\le \mathrm{H}^{-1}\left(\max_{r\ge i}\max_{s\ge r}\beta_{rs}\right)\right)\right\}\vee\mu\left\{\bigcup_{j=1}^{\infty}\left(\delta_j\le G^{-1}\left(\max_{r\ge 1}\max_{s\ge r\vee j}\beta_{rs}\right)\right)\right\}\vee\mu\left\{\bigcup_{i=2}^{\infty}\left(\alpha_i>\mathrm{F}^{-1}\left(1-\max_{r\ge i-1}\max_{s\ge r}\beta_{rs}\right)\right)\right\}$$

$$=\left(\bigvee_{i=1}^{\infty}\mu\left\{a_i\le \mathrm{H}^{-1}\left(\max_{r\ge i}\max_{s\ge r}\beta_{rs}\right)\right\}\right)\vee\left(\bigvee_{j=1}^{\infty}\mu\left\{\delta_j\le G^{-1}\left(\max_{r\ge 1}\max_{s\ge r\vee j}\beta_{rs}\right)\right\}\right)\vee\left(\bigvee_{i=2}^{\infty}\mu\left\{\alpha_i>\mathrm{F}^{-1}\left(1-\max_{r\ge i-1}\max_{s\ge r}\beta_{rs}\right)\right\}\right)$$

$$\le\left(\bigvee_{i=1}^{\infty}\max_{r\ge i}\max_{s\ge r}\beta_{rs}\right)\vee\left(\bigvee_{j=1}^{\infty}\max_{r\ge 1}\max_{s\ge r\vee j}\beta_{rs}\right)\vee\left(\bigvee_{i=2}^{\infty}\max_{r\ge i-1}\max_{s\ge r}\beta_{rs}\right)$$

$$=\left(\max_{r\ge 1}\max_{s\ge r}\beta_{rs}\right)\vee\left(\max_{r\ge 1}\max_{s\ge r}\beta_{rs}\right)\vee\left(\max_{r\ge 1}\max_{s\ge r}\beta_{rs}\right)$$

$$=\left(\max_{r\ge 1}\max_{s\ge r}\beta_{rs}\right)$$

Consequently, the theorem is verified since (5.4) holds.

$$\mathrm{H}(x)=\begin{cases}0, & if\ x<1\\ 1, & if\ x\ge 1\end{cases}$$

Remark-2: The uncertain queueing model with group arrivals becomes the uncertain single-service queueing model studied by Yao [27] when the unknown group sizes a1,a2,... are always constant 1. In this instance, we have

and then we can derive that the uncertainty distribution of busy period is

$$Q(t)=\max_{r\ge 1}\max_{s\ge r}\sup_{x\le t}\mathrm{H}\left(\frac{s}{r}\right)\wedge\left(1-\mathrm{F}\left(\frac{x}{r}\right)\right)\wedge G\left(\frac{x}{s}\right)$$

$$=\max_{r\ge 1}\max_{s\ge r}\sup_{x\le t}\left(1-\mathrm{F}\left(\frac{x}{r}\right)\right)\wedge G\left(\frac{x}{s}\right)$$

$$=\max_{r\ge 1}\sup_{x\le t}\left(1-\mathrm{F}\left(\frac{x}{r}\right)\right)\wedge G\left(\frac{x}{s}\right),$$

which is reduced to the result of uncertain single – service queueing model derived by Yao[27].

VI. CONCLUSION

The activity index and busy time were examined in this article as the most important quantitative indicators of the uncertainty queueing system. The analytical formula and uncertainty distribution of both quantitative indicators were also derived. This paper presented an uncertain queueing model with group arrivals, where customers arrive in groups, for the purpose of modeling the complex systems that provide services. Lastly, the results of the suggested uncertain queueing model with group arrivals are demonstrated by a few examples in this paper.

REFERENCES

1. Erlang A(1909)The theory of probabilities and telephone conversations. Nyt Tidsskr Mat20(B):33–39

2. KeH,YaoK(2016) Block replacement policy in uncertain environment. Reliab Eng Syst Saf 148:119–124
3. Kendall D (1953) Stochastic processes occurring in the theory of queues and their analysis by the method of the imbedded Markov chain.AnnMathStat24(3):338–354
4. LindleyD(1952) The theory of queues with a single server. Math Proc Camb48(2):277–289
5. Lio W, Liu B (2020) Shortage index and shortage time of uncertain production risk process. IEEETrans Fuzzy Syst 28(11):2856–2863
6. Liu B(2007) Uncertainty theory,2nd edn.Springer,Berlin
7. LiuB(2008) Fuzzy process, hybrid process and uncertain process.J Uncertain Syst 2(1):3–16
8. Liu B (2009) Some research problems in uncertainty theory.J Uncertain Syst 3(1):3–10
9. Liu B (2010) Uncertainty theory:a Branch of mathematics for modeling human uncertainty. Springer, Berlin
10. LiuB(2013) Extreme value theorems of uncertain process with application to insurance risk model. Soft Comput 17(4):549–556
11. LiuY(2021) Uncertain circuit equation. J Uncertain Syst 14(3):Article2150018
12. LiuY(2022) Analysis of China'spopulation with uncertain statistics. J UncertainSyst15(4):Article2243001
13. Liu Y (2022) Moment estimation for uncertain regression model with application to factors analysis of grain yield. Commun Stat Simul C. https://doi.org/10.1080/03610918.2022.2160461
14. Liu Y (2023) Uncertain insurance risk process with multiple premium processes and single claim process. J Amb Intell Hum Comput14(8):10027–10038
15. Liu Y, Liu B (2021) Waiting time and idle time of uncertain queueing system.Int Jgen Syst50(8):871–890
16. LiuY,LiuB(2022) Residual analysis and parameter estimation of uncertain differential equations. Fuzzy Optim Decis Mak21(4):513–530
17. Liu Y, Liu B (2023) A modified uncertain maximum likelihood estimation with applications in uncertain statistics. Commun Stat TheorMethods. https://doi.org/10.1080/03610926.2023.2248534
18. LiuY,LiuB(2023) Estimation of uncertainty distribution function by the principle of least squares. Commun Stat Theor Methods.https://doi.org/10.1080/03610926.2023.2269451
19. Liu Z, Yang Y (2020) Uncertain insurance risk process with multiple classes of claims.ApplMathModel83:660–673
20. Liu Z, Yang X (2021) Uncertain insurance risk process with single premium and multiple classes of claims. J Amb Intell Hum Comput12:7685–7702
21. Liu Y, Zhou L (2021) Modeling RL electrical circuit by multi factor uncertain differential equation.Symmetry 13(11):Article2103
22. Pollaczek F(1930) Über eineaufgabeder wahrscheinlichkeitstheorie. MathZ32(1):65–100
23. Pollaczek F (1934) Über das warteproblem. Math Z 38(1):492–537
24. PollaczekF(1957) Problèm esstochastiques posésparlephenomena deformationd'unequeued'attente àunguichetetpardesphénomènesapparentés.MemSci Math136:1–122
25. Qin Z (2009) On lognormal uncertain variable. In: Proceedings of the 8th international conference information and management sciences (Kunming,China),pp753–755
26. Yang L, Liu Y (2023) Solution method and parameter estimation of uncertain partial differential equation with application to China'spopulation.Fuzzy Optim Decis Mak 23(1):155–177.
27. YaoK(2021) Anun certain single-server queueing model.J UncertainSyst14(1):Article2150001
28. Yao K, Li X (2012) Uncertain alternating renewal process and its application.IEEETransFuzzySyst20(6):1154–1160
29. YaoK,QinZ(2015) Amodified insurance risk process with uncertainty. Insur Math Econ 62:227–233
30. YaoK,RalescuD(2013) Age replacement policy in uncertain environment.Iran J Fuzzy Syst 10(2):29–39

31. YaoK,ZhouJ(2018) Ru in time of uncertain insurance risk process. IEEETrans Fuzzy Syst26(1):19–28
32. Zhang X, Ning Y, Meng G (2013) Delayed renewal process with uncertain inter arrival times. Fuzzy Optim Decis Making 12(1):79–87