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A Queueing System with Servers Disguised as Customers

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Abstract- Customer service has been modeled and optimized in many real-world scenarios using queuing theory. In this study, we suggest a new model that is inspired by the occurrence of pseudo progression in cancer, where the length of a line appears to expand for a while before decreasing more quickly. We consider servers to be "disguised" as customers when they join the line. We use matrix analytic techniques to determine the general equations for this model and use numerical simulations to show how it functions.

Keywords— queueing model, cancer, matrix

I. INTRODUCTION

Numerous queueing model variations that account for heterogeneous servers, various customer kinds, and priority queues have been put forth. We present a novel queueing paradigm in this work, where servers show up posing as clients. This model was inspired by the documentary "Jim Allison: Breakthrough" (Haney, 2020), which discussed the false progression phenomena in cancer immunotherapy treatments. A deceptive increase in tumor size or appearance in new lesions followed by a reduction in tumor burden is known as pseudo progression. Treatment causes the apparent tumor size to expand, which is the cause of the deceptive increase. Jia et al. (2019) and Ma et al. (2019) provide a comprehensive analysis of pseudo progression and compare it to actual tumor progression. Similarly, the (hidden) entrance of servers among the consumers appears to cause our suggested queueing model to grow in size for a while. The queue size (rapidly) drops after these covert servers start to serve. One way to think of the addition of servers to the queue under the garb of consumers is as an addition of heterogeneous customers to the systemThe literature on queues with heterogeneous clients and servers, as well as matrix geometric solutions for queueing theory, are reviewed in section (2). In section (3), we derive the model's equations and present the model's phase diagram and notation. In Section (4), we use matrix analytic techniques to do numerical simulations for the most basic example of our model and show how the queue attributes vary for various parameter values. We offer closing thoughts in section (5).

II. LITERATURE REVIEW

Section (2) reviews the literature on queues with heterogeneous clients and servers as well as matrix geometric solutions for queueing theory. The model's equations, phase diagram, and notation are presented in section (3). In Section (4), we perform numerical simulations for our model's most basic example using matrix analytic techniques to demonstrate how the queue properties change for different parameter values. In section (5), we provide concluding remarks. A queueing system with servers that show up when the number of clients above a certain threshold was introduced by Gurtov and Mazalov (2012). A line with diverse arrivals and service periods was proposed by Yechiali and Naor (1971). Arrival intensities can "jump" between two distinct levels with varying rates in their model. Each level's duration is dispersed exponentially. In order to reduce a customer's waiting time in the situation of priority queueing, Kim et al. (2011) investigate the issue of allocating consumers to various servers (with varying service rates). We employ a model by Li and Yang (2000) where the number of servers in the queue is dynamic and contingent on the number of customers. A search and release procedure is used to add or remove servers. For our model, we employ matrix analytic techniques, which are reviewed by Neuts (1984). Nelson (1991) and Stewart (2021) provide a method for solving such systems.

III. MODEL

1. Notation

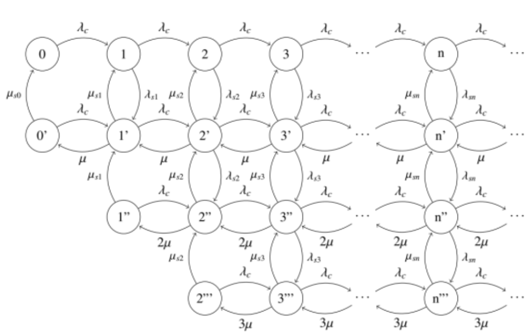
We first teach some basic notation before defining the model:

Clients are served on a first-come, first-served basis and enter the system in accordance with a homogeneous Poisson process with rate λc. Service times have a rate of μ and are independent, exponential random variables. In our concept, servers have a rate of λsn when they enter the system with consumers and a rate of μsn when they exit. In general, we suppose that the number of customers (n) in the system can affect the arrival and departure of servers; for instance, a new server would come more quickly if there were more consumers.

{0, 1, 2,...k} is the number of servers. To keep things simple, we derive the equations for K = 3 servers and limit the number of servers in the system to K. There are 0, 1, 2, or 3 servers in the system for any number of n.▪ {0, 1, 2, 3,...} is the number of consumers and k ▪ {0, 1, 2,...k} The system's states are represented by (n, k), where n.

2. Phase Diagram

For every state (n, k), k is referred to as the inter level state and n as the level state. We have four inter level states for every level. The process's phase diagram is shown below:



The number of servers is shown in the above diagram by the dashes next to the numbers. The shift from (1, 2) to (1, 1) shows that we assume a server departs as soon as it is no longer required. Since a transition cannot take place from (1, 2) to (0, 2), we do not have the states (0, 2). The time-dependent probability equations for the model will now be derived.

3. The Model's Matrix Analytic Solution

The equations for the model's most general example are now derived. The system's generator matrix is provided by:

$$Q=\begin{pmatrix} -\lambda_c & \lambda_c & 0 & 0 & 0 & 0 & 0 & \dots \\ -\mu_s & -\mu_s-\lambda_c & 0 & \lambda_c & 0 & 0 & 0 & \dots \\ 0 & 0 & -\lambda_{s1}-\lambda_c & -\lambda_{s1} & 0 & \lambda_c & 0 & \dots \\ 0 & \mu & \mu_{s1} & -\mu_{s1}-\mu-\lambda_c & 0 & 0 & \lambda_c & \dots \\ 0 & 0 & 0 & \mu_{s1} & -\mu_{s1}-\lambda_c & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

The following is how we can express Q as a block matrix:

$$Q=\begin{pmatrix} B_{00} & B_{01} & 0 & 0 & 0 & 0 & 0 & \dots \\ B_{10} & B_{11} & B_{12} & 0 & 0 & 0 & 0 & \dots \\ 0 & B_{21} & B_{22} & A_1 & 0 & 0 & 0 & \dots \\ 0 & 0 & A_0 & A_3 & A_1 & 0 & 0 & \dots \\ 0 & 0 & 0 & A_0 & A_4 & A_1 & 0 & \dots \\ 0 & 0 & 0 & 0 & A_0 & A_5 & A_1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{1}$$

$$B_{00}=\begin{bmatrix} -\lambda_c & \lambda_c \\ \mu_{s0} & -\mu_{s0}-\lambda_c \end{bmatrix}$$

$$B_{01}=\begin{bmatrix} 0 & 0 & 0 \\ 0 & \lambda_0 & 0 \end{bmatrix}$$

$$B_{11}=\begin{bmatrix} 1 & 0 \\ 0 & \mu \\ 0 & 0 \end{bmatrix}$$

$$B_{11}=\begin{bmatrix}-\hat{\lambda} & \lambda_{s1} & 0\\ \mu_{s1} & -\mu-\mu_{s1}-\lambda_c-\lambda_{s1} & 0\\ 0 & \mu_{s1} & -\mu_{s1}-\lambda_c\end{bmatrix}$$

$$B_{12}=\begin{bmatrix}\lambda_c & 0 & 0 & 0\\ 0 & \lambda_c & 0 & 0\\ 0 & 0 & \lambda_c & 0\end{bmatrix}$$

$$B_{21}=\begin{bmatrix}0 & 0 & 0\\ 0 & \mu & 0\\ 0 & 0 & 2\mu\\ 0 & 0 & 0\end{bmatrix}$$

$$B_{22}=\begin{bmatrix}-\hat{\lambda} & \lambda_{s2} & 0 & 0\\ \mu_{s2} & -\mu_{s2}-\mu-\hat{\lambda} & \lambda_{s2} & 0\\ 0 & \mu_{s2} & -\mu_{s2}-2\mu-\lambda_{s2}-\lambda_c & 0\\ 0 & 0 & \mu_{s2} & -\mu_{s2}-\lambda_c\end{bmatrix} \quad (2)$$

are referred to as the border conditions. Additionally, we have the matrices:

$$A_1=\begin{bmatrix}\lambda_c & 0 & 0 & 0\\ 0 & \lambda_c & 0 & 0\\ 0 & 0 & \lambda_c & 0\\ 0 & 0 & 0 & \lambda_c\end{bmatrix}$$

, $$A_0=\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & \mu & 0 & 0\\ 0 & 0 & 2\mu & 0\\ 0 & 0 & 0 & 3\mu\end{bmatrix} \quad (3)$$

and

$$A_n=\begin{bmatrix}-\lambda_{sn}-\lambda_c & \lambda_{ssn} & 0 & 0\\ \mu_{sn} & -\mu-\mu_{sn}-\lambda_{sn}-\lambda_c & \lambda_{sn} & 0\\ 0 & \mu_{sn} & -2\mu-\mu_{sn}-\lambda_{sn}-\lambda_c & \lambda_{sn}\\ 0 & 0 & \mu_{sn} & -3\mu-\mu_{sn}-\lambda_c\end{bmatrix}$$

Where n ▪ 3. Observe that the travel to a lower level is represented by matrix A0, while the movement to a higher level is represented by matrix A1.

Let ▪ = (▪0, ▪1, ▪2, ...) represent the system's limiting distribution, where ▪i represents the ith level's limiting distribution. Next, we have the following equations:

$\Pi_0 = 0$

and (4)

$\Pi_{\mathbf{e}} = 0$ (5)

Where e stands for a vector of 1′ s in a column. We obtain a set of boundary conditions by solving the two equations above:

$$\pi_0 B_{00}+\pi_1 B_{10}=0$$

$$\pi_0 A_1+\pi_1 B_{11}+\pi_2 B_{21}=0 \quad (6)$$

$$\pi_1 A_1+\pi_2 B_{22}+\pi_3 A_0=0$$

The recursive relation for i ▪ 3 is an follows

$$\pi_{i-1} A_1+\pi_i A_i+\pi_{i+1} A_0=0 \quad (7)$$

In order to answer the aforementioned equations, we must express An in terms of the known values. We consider the most basic scenario in the part that follows, which is when ▪sn = ▪s and ▪sn = ▪s and determine the appropriate queue's characteristics.
Simplest Case: ▪sn = ▪s and ▪sn = ▪s

In this instance, An is a constant matrix that is provided by:

$$B_{22}=\begin{bmatrix}-\hat{\lambda} & \lambda_s & 0 & 0\\ \mu_s & -\mu-\mu_s-\hat{\lambda} & \lambda_s & 0\\ 0 & \mu_s & -2\mu-\mu_s-\hat{\lambda} & \lambda_s\\ 0 & 0 & \mu_s & -3\mu-\mu_s-\lambda_c\end{bmatrix} \quad (8)$$

Where . Our generator matrix becomes:

$$Q=\begin{pmatrix} B_{00} & B_{01} & 0 & 0 & 0 & 0 & 0 & \dots \\ B_{10} & B_{11} & B_{12} & 0 & 0 & 0 & 0 & \dots \\ 0 & B_{21} & B_{22} & A_1 & 0 & 0 & 0 & \dots \\ 0 & 0 & A_0 & A_2 & A_1 & 0 & 0 & \dots \\ 0 & 0 & 0 & A_0 & A_2 & A_1 & 0 & \dots \\ 0 & 0 & 0 & 0 & A_0 & A_2 & A_1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

(9)

The Equation (7) for the recurrence relation simplifies to:

$$\pi_{i-1}A_1 + \pi_i A_2 + \pi_{i+1}A_0 = 0 \quad (10)$$

The limiting distributions can now be found (n ・ 3),Observe that the matrix Q begins to repeat exactly like a quasi-birth-and-death process when we get to the third level.

4. Stability Condition

Assume that the generating matrix A is provided by A = A0 + A1 + A2. In the system's repeating part, this serves as the generator. We first determine the stationary distribution for A in order to formulate a criterion for the queue's stability,

given by $\pi_A = (\pi_{A0}, \pi_{A1}, \pi_{A2}, \pi_{A3})$ We employ the following formulae for this:

$$\pi_A A = 0$$

And

$$\sum_{i=0}^{3} \pi_{Ai} = 1$$

Then, we have:

$$(\pi_{A0}, \pi_{A1}, \pi_{A2}, \pi_{A3}) \cdot \begin{pmatrix} -\lambda_s & \lambda_s & 0 & 0 \\ \mu_s & -\mu_s - \lambda_s & \lambda_s & 0 \\ 0 & \mu_s & \mu_s - \lambda_s & \lambda_s \\ 0 & 0 & \mu_s & -\mu_s \end{pmatrix} = 0$$

(11)

Solving for , $(\pi_{A0}, \pi_{A1}, \pi_{A2}, \pi_{A3})$

$$-\lambda_s \pi_{A0} + \mu_s \pi_{A1} = 0$$

$$\Rightarrow \pi_{A1} = \rho_s \pi_{A0}$$

$$\lambda_s - (\mu_s + \lambda_s)\pi_{A1} + \mu_3 \pi_{A2} = 0 \quad (12)$$

$$\Rightarrow \pi_{A2} = \rho_s \pi_{A1} = \rho_s^2 \pi_{A0}$$

$$\lambda_s \pi_{A1} - (\mu_s + \lambda_s)\pi_{A2} + \mu_s \pi_{A3} = 0$$

$$\Rightarrow \pi_{A3} = \rho_s \pi_{A2} = \rho_s^3 \pi_{A0}$$

We need the drift to lower levels to be larger than the drift to higher levels for ergodicity. That is, the system will be more inclined to go to lower level.

$$\pi_A A_1 \mathrm{e} < \pi_A A_0 \mathrm{e}$$

$$\pi_A \cdot \begin{bmatrix} \lambda_c & 0 & 0 & 0 \\ 0 & \lambda_c & 0 & 0 \\ 0 & 0 & \lambda_c & 0 \\ 0 & 0 & 0 & \lambda_c \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} < \pi A \cdot \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \mu & 0 & 0 \\ 0 & 0 & 2\mu & 0 \\ 0 & 0 & 0 & 3\mu \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

(14)

$$\Rightarrow \lambda_c \sum_{i=0}^{3} \pi_{Ai} < (0, \mu\pi_{A1}, 2\mu\pi_{A2}, 3\mu\pi_{A3}) \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

The gives us our ergodicity condition:

$$\rho_c < \frac{\rho_s(1 + 2\rho_s + 3\rho_s^2)}{(1+\rho_s)(1+\rho_s^2)}$$

where $\rho_c = \lambda_c / \rho_c$ (15)

5. Limiting Distribution Solution

After determining whether the queue is stable for the specified values of $\lambda_c, \mu, \lambda_s$ and μ_s , The limiting distribution can be found. Given that A0, A1, and A2 are all constant matrices, Equation (10) We can anticipate that for $i \geq 3, \pi_i = \pi_{i-1}R$, where R is a matrix of constants. When we repeatedly substitute in the recurrence relation, we obtain:

$$\pi_i = \pi_2 R^{i-2}, \; i \geq 3 \quad (16)$$

The simplest method to solve for R is to iterate numerically until the elements of the matrix differ by less than a specified value. Substituting the value of π_{i-1}, π_i and π_{i+1}, the iterative formula is found as follows:

$$R^2 A_0 + RA_2 + A_1 = 0$$

$$\Rightarrow R^2 A_0 A_2^{-1} + R + A_1 A_2^{-1} = 0$$

$$R = -A_1 A_2^{-1} - R^2 A_0 A_2^{-1} = -V - R^2 W \text{ where}$$

$$R_{(0)} = 0 \;\; R_{(k+1)} = -V - R_{(k)}^2 W, \;\; k = 0, 1, 2, \ldots$$

The boundary equation (6) is now solved for π_0, π_1 and π_2. Substituting the value of $\pi_3 = \pi_2 R$, The equations can be expressed as follows in matrix form:

$$(\pi_0, \pi_1, \pi_2) \begin{pmatrix} B_{00} & B_{01} & 0 \\ B_{10} & B_{11} & B_{12} \\ 0 & B_{21} & B_{22} + RA_0 \end{pmatrix} = (0,0 \,|\, 0,0,0 \,|\, 0,0,0,0)$$

Note that $\pi_0 = (\pi_{00}, \pi_{01}), \pi_1 = (\pi_{10}, \pi_{11}, \pi_{12})$ and $\pi_{12} = (\pi_{20}, \pi_{21}, \pi_{22}, \pi_{23})$. The equation above can be rewritten as follows:

$$\begin{pmatrix} B_{00}^T & B_{10}^T & 0 \\ B_{01}^T & B_{11}^T & B_{21}^T \\ 0 & B_{12}^T & (B_{22} + RA_0)^T \end{pmatrix} \begin{pmatrix} \pi_0^T \\ \pi_1^T \\ \pi_2^T \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \bar{0} \\ 0 \\ 0 \\ \bar{0} \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (17)$$

Once we have values for ▪0, ▪1 and ▪2, we normalize them to get probabilities:

$$\pi e = 1$$

$$\alpha = \pi_0 e + \pi_1 e + \pi_2 e + \sum_{i=1}^{\infty} \pi_2 R^i$$

$$\Rightarrow \alpha = \pi_0 e + \pi_1 e + \sum_{i=0}^{\infty} \pi_2 R^i$$

$$\alpha = \pi_0 e + \pi_1 e + \pi_2 (I - R)^{-1} \quad (18)$$

In the above equations, we take e merely to be a vector of ones of required dimension for each term. After obtaining the boundary limiting distributions, we can identify any . In the next section, we find the properties of the queue.

6. Properties of the Queue

Expected length of queue: Since an observer of the queue cannot tell a disguised server from a customer, the estimated queue length for our model includes the number of servers. Prior to determining the predicted queue length expression, we first define:

$$v_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad v_1 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \qquad v = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}$$

The expected queue length expression will then be provided by:

$$E(L) = \sum_{n=0}^{\infty} \sum_{k} (n+k) p_{(n,k)}$$

$$= \pi_0 v_0 + \pi_1 (e_1 + v_1) + \sum_{n=2}^{\infty} \sum_{k} (n+k) p_{(n,k)}$$

$$= \pi_0 v_0 + \pi_1 (e_1 + v_1) + \sum_{n=2}^{\infty} \pi_2 R^{n-2} e + \sum_{n=2}^{\infty} \pi_2 R^{n-2} v$$

$$= \pi_0 v_0 + \pi_1(e_1 + v_1) + \pi_2\left(\sum_{n=2}^{\infty} nR^n\right)e + \pi_2\left(\sum_{n=2}^{\infty} R^n\right)v$$

Simplifying, we obtain our final equation:

$$E(L) = \pi_0 v_0 + \pi_1(e_1 + v_1) + \pi_2[R(I-R)^2 + 2(I-R)^{-1}]e + \pi_2(I-R)^{-1}v \tag{19}$$

$$E(L_n) = \sum_{n=1}\sum_{k}(n-k)p(n,k)$$

$$= \sum_{n=1}^{3}\sum_{k=0}^{n}(n-k)p(n,k) + \sum_{n=4}^{\infty}\sum_{k=0}^{3}(n-k)p(n,k)$$

$$= \sum_{n=1}^{3}\sum_{k=0}^{n}(n-k)p(n,k) + \sum_{n=4}^{\infty}\sum_{k=0}^{3}np(n,k) - \sum_{n=4}^{\infty}\sum_{k=0}^{3}kp(n,k)$$

$$= \sum_{n=1}^{3}\sum_{k=0}^{n}(n-k)p(n,k) + \pi_2\left(\sum_{n=4}^{\infty}nR^{n-2}e - \sum_{n=4}^{\infty}R^{n-2}v\right)$$

We get the following by rearranging the terms and modifying the limits:

$$E(L_n) = \pi_{10} + (2\pi_{20} + \pi_{21}) + (3\pi_{30} + 2\pi_{31} + \pi_{32}) + \pi_2[R(I-R)^{-2}$$
$$+ 2(I-R)^{-1} - (2I + 3R)]e - \pi_2[(I-R)^{-1} - (I+R)]v \tag{20}$$

By applying Little's Law, the anticipated waiting time can be determined accordingly:

$$E(L_n) = \lambda_c E(W) \Rightarrow E(W) = \frac{E(L_n)}{\lambda_c}.$$

Probability of Delay: Delay probability, or the likelihood that a customer will have to wait, is represented as ▪w, and the following is how it is derived:

$$\Pi_w = \sum_{k}\sum_{n=k+1}^{\infty}p(n,k) = \sum_{n=1}^{3}\sum_{k=0}^{n}p(n,k) + \sum_{n=4}^{\infty}\pi_n e \tag{21}$$

$$= (\pi_{10} + \pi_{11}) + (\pi_{20} + \pi_{21} + \pi_{22}) + \pi_2((I-R) - 1 - I)e$$

7. Simulations with numbers

We simulate the model for various values of λ_c, λ_s, μ and μ. The model's output is shown

$\lambda_c = 1$, $\lambda_s = 2$					
μ	μ_s	E(L)	E(L_n)	W	Π_w
1	1	2.609	0.287	2.609	0.399
1	2	1.533	0.426	1.533	0.375
1	3	1.166	0.498	1.166	0.339
2	1	1.478	0.204	1.478	0.282
2	2	0.894	0.270	0.894	0.250
2	3	0.710	0.302	0.710	0.225
3	1	1.042	0.150	1.042	0.213
3	2	0.669	0.193	0.669	0.186
3	3	0.543	0.214	0.543	0.168
$\lambda_c = 2$, $\lambda_s = 1$					
μ	μ_s	E(L)	E(L_n)	W	Π_w
1	1	4.474	0.756	0.378	0.557
1	2	4.055	1.627	0.813	0.657
1	3	4.152	2.450	1.250	0.713
2	1	3.559	0.764	0.382	0.490
2	2	3.211	1.493	0.747	0.579
2	3	3.282	2.096	1.048	0.621
3	1	2.934	0.732	0.366	0.440
3	2	2.658	1.311	0.655	0.509
3	3	2.709	1.757	0.879	0.543

From the values given, we may draw the following conclusions:

- The values of E(Ln) drop as μ rises for a given μs. In other words, the number of clients in the system decreases as the service rate rises. Naturally, when μ increases, the estimated waiting time E(W) and the delay probability Πw will likewise decrease.
- For a fixed μ, the increase of μs resulted in an increase of customers waiting in the queue (E(Ln)). This is consistent with the assumption that as servers leave the queue at faster rates, more customers will be left waiting. An interesting remark is that when λc < λs, the delay probability and waiting time decrease even if μs increases. This could be attributed to the fact that though the rate of departure of servers increases, their rate of arrival makes up for it. Notice that when λc > λs, the expected waiting time and delay probability increase with μs.
- Customers waiting in the queue increased as μs increased for a fixed μ E(Ln). This is in line with the theory that more clients will be left waiting as servers depart the line more quickly. Interestingly, even if μs grows, the delay

probability and waiting time decrease when $\lambda c < \lambda s$. This may be explained by the fact that, despite an increase in servers' departure rate, their arrival rate compensates for it. The estimated waiting time and delay probability increase with μs when $\lambda c > \lambda s$.

Example 1: $\lambda c = 1$; $\mu = 2$; $\lambda s = 3$; $\mu s = 4$:
Here, $\rho c = 0.5$; $\rho s = 0.75$ and $(\rho s(1 + 2\rho s + 3\rho 2s)) / ((1 + \rho s)(1 + \rho 2s)) = 1.149$, Therefore, the queue will be stable since the ergodicity criterion is met,

$$B_{00}=\begin{bmatrix}-1 & 1\\ 4 & -5\end{bmatrix} \qquad B_{01}=\begin{bmatrix}0 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}$$

$$B_{10}=\begin{bmatrix}1 & 0\\ 0 & 2\\ 0 & 0\end{bmatrix} \qquad B_{11}=\begin{bmatrix}-4 & 3 & 0\\ 4 & -7 & 0\\ 0 & 4 & -5\end{bmatrix}$$

$$B_{12}=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix}$$

$$B_{21}=\begin{bmatrix}0 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 4\\ 0 & 0 & 0\end{bmatrix}$$

$$B_{22}=\begin{bmatrix}-4 & 3 & 0 & 0\\ 4 & -10 & 3 & 0\\ 0 & 4 & -12 & 3\\ 0 & 0 & 4 & -5\end{bmatrix}$$

$$A_0=\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 2 & 0 & 0\\ 0 & 0 & 4 & 0\\ 0 & 0 & 0 & 6\end{bmatrix} A_1=I$$

$$A_2=\begin{bmatrix}-4 & 3 & 0 & 0\\ 4 & -10 & 3 & 0\\ 0 & 4 & -12 & 3\\ 0 & 0 & 4 & -11\end{bmatrix}$$

Applying the Equation (17), we get $\pi A = (0.3666; 0.2743; 0.2066; 0.1543)$ with condition (19) signifying the stability of the system. Until the difference between elements of subsequent rounds is smaller than 10-12, we found R iteratively. R was thus given as:

$\pi_0 = (0.6557041, 0.1519753)$
$\pi_1 = (0.052086275, 0.052086275, 0.005271742)$
$\pi_2 = (0.030662973, 0.017641404, 0.006589678, 0.004503371)$

Once we have ▪0, ▪1 and ▪2,We are able to determine the system's attributes and all other probabilities:

- The system's overall anticipated queue length E(L) = 0.5764801741.
- A customer's anticipated wait time W = 0.5764801741.
- The anticipated length of the line (customers only) E(Ln) = 0.2068827196.
- The likelihood that a new customer will have to wait ▪w = 0.1825454937.

Example 2: $\lambda c = 2$; $\mu = 1$; $\lambda s = 1$; $\mu s = 2$:
Here, $\rho c = 2$; $\rho s = 0.5$ and .

IV. CONCLUSIONS

In this study, we introduced a new queueing system where servers join customers in line. The line first appears to get longer because an observer cannot tell the difference between a server and a customer. The wait lengthens as the new servers begin to service the clients. This model's equations are developed, and numerical simulations are used to illustrate the model.

Drawing on the parallels with the original motive, pseudo progression, the model can be further expanded. We might assume, for instance, that there is a preemptive line with a variety of customer kinds with varying arrival and service speeds. The number of clients in the system may affect the arrival and/or service rates; if the number of clients rises, servers with quicker service rates may show up.

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