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Unique Common Fixed Point of Multivalued Generalized φ-Weak Contractive Mappings with Integral Type Inequality

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Abstract

The aim of this paper is to establish a unique Common Fixed Point of Multivalued Generalized φ-Weak Contractive Mappings in Integral Type Inequality. This result improves the result of Nadler and Daffer–Kaneko's and references there in.

Key words: Common fixed point, contractive mapping, fixed point.

1 Introduction

Fixed point and coincidence results are presented for multivalued generalized φ-weak contractive mappings on complete metric spaces, where $\varphi\colon [0,+\infty) \to [0,+\infty)$ is a lower semicontinuous function with $\varphi(0) = 0$ and $\varphi(t) > 0$ for all $t > 0$. Our results extend previous results by Zhang and Song (2009), as well as by Rhoades (2001), Nadler (1969), and Daffer and Kaneko (1995).

Theorem : Let (X, d) be a metric space. We denote the family of all nonempty closed and bounded subsets of X by $CB(X)$.

A mapping $T\colon X \to X$ is said to be φ-weak contractive if there exists a map $\varphi\colon [0,+\infty) \to [0,+\infty)$ with $\varphi(0) = 0$ and $\varphi(t) > 0$ for all $t > 0$, such that

$$d(Tx, Sy) \le M(x, y) - \varphi(M(x, y)) \tag{1}$$

for all $x, y \in X$.

Also two mappings $T, S\colon X \to X$ are called generalized φ-weak contractions if there exists a map $\varphi\colon [0,+\infty) \to [0,+\infty)$ with $\varphi(0) = 0$ and $\varphi(t) > 0$ for all $t > 0$, such that

$$d(Tx, Sy) \le M(x, y) - \varphi(M(x, y)) \tag{2}$$

for all $x, y \in X$, where

$$M(x,y) := \max\Big\{d(x,y),\, d(x,Tx),\, d(y,Ty),\, \tfrac{1}{2}\big[d(x,Sy) + d(y,Tx)\big]\Big\}. \tag{3}$$

A mapping $T\colon X \to CB(X)$ is said to be a weak contraction if there exists $0 \le \alpha < 1$ such that

$$H(Tx, Ty) \le \alpha N(x, y), \tag{4}$$

for all $x, y \in X$, where H denotes the Hausdorff metric on $CB(X)$ induced by d, that is,

$$H(A,B) = \max\Big\{\sup_{x\in A} d(x,B),\, \sup_{y\in B} d(y,A)\Big\} \tag{5}$$

for all $A, B \in CB(X)$, and where

$$N(x,y) := \max\Big\{d(x,y),\, d(x,Tx),\, d(y,Ty),\, \tfrac{1}{2}\big[d(x,Ty) + d(y,Tx)\big]\Big\}. \tag{6}$$

A mapping $T\colon X \to CB(X)$ is said to be φ-weak contractive if there exists a map $\varphi\colon [0,+\infty) \to [0,+\infty)$ with $\varphi(0) = 0$ and $\varphi(t) > 0$ for all $t > 0$, such that

$$d(Tx, Ty) \le d(x, y) - \varphi(d(x, y)) \tag{7}$$

for all $x, y \in X$.

The concepts of weak and φ-weak contractive mappings were defined by Daffer and Kaneko in 1995. Many authors have studied fixed points for multivalued mappings. Among many others, see, for example, 1–4, and the references therein.

In the following theorem, Nadler [3] extended the Banach Contraction Principle to multivalued mappings.

Theorem 1.1. *Let (X, d) be a complete metric space. Suppose $T\colon X \to CB(X)$ is a contraction mapping in the sense that for some $0 \le \alpha < 1$,*

$$\int_0^{H(Tx,Ty)} \varphi(t)\,dt \le \alpha \int_0^{d(x,y)} \varphi(t)\,dt, \tag{8}$$

for all $x, y \in X$. Then there exists a point $x \in X$ such that $x \in Tx$ (i.e., x is a fixed point of T).

Daffer and Kaneko [1] proved the existence of a fixed point for a multivalued weak contraction mapping of a complete metric space X into $CB(X)$.

Theorem 1.2. *Let (X, d) be a complete metric space and $T\colon X \to CB(X)$ be such that*

$$\int_0^{H(Tx,Ty)} \varphi(t)\,dt \le \alpha \int_0^{N(x,y)} \varphi(t)\,dt, \tag{9}$$

for some $0 \le \alpha < 1$ and for all $x, y \in X$ (i.e., weak contraction). If $x \to d(x, Tx)$ is lower semicontinuous (l.s.c.), then there exists $x_0 \in X$ such that $x_0 \in Tx_0$.

Rhoades [5, Theorem 2] proved the following fixed point theorem for φ-weak contractive single valued mappings, giving another generalization of the Banach Contraction Principle.

Theorem 1.3. *Let (X,d) be a complete metric space and $T\colon X \to CB(X)$ be a mapping such that*

$$\int_0^{d(Tx,Ty)} \varphi(t)\,dt \le \int_0^{d(x,y)} \varphi(t)\,dt - \varphi\left(\int_0^{d(x,y)} \varphi(t)\,dt\right), \tag{10}$$

for every $x,y \in X$ (i.e., weak contraction), where $\varphi\colon [0,+\infty) \to [0,+\infty)$ is a continuous and nondecreasing function with $\varphi(0) = 0$ and $\varphi(t) > 0$ for all $t > 0$. Then T has a unique fixed point.

By choosing $\psi(t) = t - \varphi(t)$, φ-weak contractions become mappings of Boyd and Wong type [6], and by defining $k(t) = (t-\varphi(t))/t$ for $t > 0$ and $k(0) = 0$, then φ-weak contractions become mappings of Reich type [7].

Recently Zhang and Song [8] proved the following theorem on the existence of a common fixed point for two single valued generalized φ-weak contraction mappings.

Theorem 1.4. *Let (X,d) be a complete metric space and $T\colon X \to CB(X)$ be a two mapping such that for all $x,y \in X$,*

$$\int_0^{d(Tx,Sy)} \varphi(t)\,dt \le \int_0^{M(x,y)} \varphi(t)\,dt - \varphi\left(\int_0^{M(x,y)} \varphi(t)\,dt\right), \tag{11}$$

(i.e., generalized φ-weak contractions), where $\varphi\colon [0,+\infty) \to [0,+\infty)$ is an l.s.c. function with $\varphi(0) = 0$ and $\varphi(t) > 0$ for all $t > 0$. Then there exists a unique point $x \in X$ such that $x = Tx = Sx$.

In Section 4, we extend Theorem 1.3 by assuming φ to be only l.s.c., and extend Theorem 1.4 to multivalued mappings.

2 Preliminaries

In this paper, (X,d) denotes a complete metric space and H denotes the Hausdorff metric on $CB(X)$.

Definition 2.1. *Two mappings $T,S\colon X \to CB(X)$ are called generalized weak contractions if there exists $0 \le \alpha < 1$ such that*

$$\int_0^{H(Tx,Sy)} \varphi(t)\,dt \le \alpha \int_0^{M(x,y)} \varphi(t)\,dt, \tag{12}$$

for all $x,y \in X$.

Definition 2.2. *Two mappings $T,S\colon X \to CB(X)$ are called generalized φ-weak contractive if there exists a map $\varphi\colon [0,+\infty) \to [0,+\infty)$ with $\varphi(0) = 0$ and $\varphi(t) > 0$ for all $t > 0$, such that*

$$\int_0^{H(Tx,Sy)} \varphi(t)\,dt \le \int_0^{M(x,y)} \varphi(t)\,dt - \varphi\left(\int_0^{M(x,y)} \varphi(t)\,dt\right) \tag{13}$$

for all $x,y \in X$.

In the proof of our main results, we will use the following well-known lemma, and refer to Nadler [3] or Assad and Kirk [9] for its proof.

Lemma 2.3. *If $A,B \in CB(X)$ and $a \in A$, then for each $\varepsilon > 0$, there exists $b \in B$ such that*

$$d(a,b) \le \alpha H(A,B) + \varepsilon \tag{14}$$

3 Extension of Nadler and Daffer–Kaneko's Theorems

The following theorem extends Nadler and Daffer–Kaneko's Theorems to a coincidence theorem, without assuming $x \to d(x,Tx)$ to be l.s.c.

Theorem 3.1. *Let (X,d) be a complete metric space, and let $T,S\colon X \to CB(X)$ be two multivalued mappings such that for all $x,y \in X$,*

$$\int_0^{H(Tx,Sy)} \varphi(t)\,dt \le \alpha \int_0^{M(x,y)} \varphi(t)\,dt, \tag{15}$$

where $0 \le \alpha < 1$ (i.e., multivalued generalized weak contractions). Then there exists a point $x \in X$ such that $x \in Tx$ and $x \in Sx$ (i.e., T and S have a common fixed point). Moreover, if either T or S is single valued, then this common fixed point is unique.

Proof. Obviously $M(x,y) = 0$ if and only if $x = y$ is a common fixed point of T and S.

Let $\varepsilon > 0$ be such that $\beta = \alpha + \varepsilon < 1$. Let exists $x_0 \in X$ such that $x_0 \in Tx_0$. By Lemma 2.3, there exists $x_1 \in Tx_1$ such that

$$\int_0^{d(x_2,x_1)} \varphi(t)\,dt \le \int_0^{H(Tx_1,Sx_0)} \varphi(t)\,dt + \varepsilon \int_0^{M(x_1,x_0)} \varphi(t)\,dt.$$

Again by using Lemma 2.3, there exists $x_3 \in Sx_2$ such that

$$\int_0^{d(x_3,x_2)} \varphi(t)\,dt \le \int_0^{H(Sx_2,Tx_1)} \varphi(t)\,dt + \varepsilon \int_0^{M(x_2,x_1)} \varphi(t)\,dt$$

By induction and using Lemma 2.3, we can find in this way a sequence $\{x_n\}$ in X such that $x_{2k+1} \in Sx_{2k}$ and

$$\int_0^{d(x_{2k+1},x_{2k})} \varphi(t)\,dt \le \int_0^{H(Sx_{2k},Tx_{2k-1})} \varphi(t)\,dt + \varepsilon \int_0^{M(x_{2k},x_{2k-1})} \varphi(t)\,dt \tag{16}$$

and $x_{2k+2} \in Tx_{2k+1}$ and

$$\int_0^{d(x_{2k+2},x_{2k+1})} \varphi(t)\,dt \le \int_0^{H(Sx_{2k+1},Tx_{2k})} \varphi(t)\,dt + \varepsilon \int_0^{M(x_{2k+1},x_{2k})} \varphi(t)\,dt \tag{17}$$

It follows that

$$\begin{aligned}
&\int_0^{d(x_{2n+1},x_{2n})} \varphi(t)\,dt \\
&\le \int_0^{H(Tx_{2n-1},Sx_{2n})} \varphi(t)\,dt + \varepsilon \int_0^{M(x_{2n-1},x_{2n})} \varphi(t)\,dt \\
&\le \beta \int_0^{M(x_{2n-1},x_{2n})} \varphi(t)\,dt \\
&= \beta \int_0^{\max\{d(x_{2n-1},x_{2n}),\, d(x_{2n-1},Tx_{2n-1}),\, d(x_{2n},Sx_{2n}),\, R_{2n}\}} \varphi(t)\,dt
\end{aligned}$$

where $R_{2n} = \frac{1}{2}[d(x_{2n-1}, Sx_{2n}) + d(x_{2n}, Tx_{2n-1})]$. Hence,

$$\begin{aligned}
&\int_0^{d(x_{2n+1},x_{2n})} \varphi(t)\,dt \\
&\le \beta \int_0^{\max\left\{d(x_{2n-1},x_{2n}),\, d(x_{2n},x_{2n+1}),\, \frac{d(x_{2n-1},x_{2n+1})}{2}\right\}} \varphi(t)\,dt \\
&= \beta \int_0^{\max\{d(x_{2n-1},x_{2n}),\, d(x_{2n},x_{2n+1})\}} \varphi(t)\,dt \\
&= \beta \int_0^{d(x_{2n-1},x_{2n})} \varphi(t)\,dt
\end{aligned}$$

since if otherwise $\int_0^{d(x_{2n},x_{2n+1})} \varphi(t)\,dt > \int_0^{d(x_{2n},x_{2n-1})} \varphi(t)\,dt$, then $\int_0^{d(x_{2n},x_{2n+1})} \varphi(t)\,dt \le \beta \int_0^{d(x_{2n},x_{2n+1})} \varphi(t)\,dt$ and so $\int_0^{d(x_{2n},x_{2n+1})} \varphi(t)\,dt = 0$. Hence $0 = \int_0^{d(x_{2n},x_{2n+1})} \varphi(t)\,dt > \int_0^{d(x_{2n},x_{2n-1})} \varphi(t)\,dt$ and this is a contradiction.

Similarly,

$$\int_0^{d(x_{2n+2},x_{2n+1})} \varphi(t)\,dt \le \beta \int_0^{d(x_{2n+1},x_{2n})} \varphi(t)\,dt \tag{18}$$

From (3.4) and (3.5), we conclude that

$$\int_0^{d(x_{k+1},x_k)} \varphi(t)\,dt \le \beta \int_0^{d(x_k,x_{k-1})} \varphi(t)\,dt \tag{19}$$

for all $k \in \mathbb{N}$. Since $\beta < 1$ and (3.6) holds, $\{x_n\}$ is a Cauchy sequence. Since (X, d) is complete, there exists $x \in X$ such that $\lim_{n\to\infty} x_n = x$.

$$\begin{aligned}
&\int_0^{d(x_{2n+2},Sx)} \varphi(t)\,dt \\
&\le \int_0^{H(Tx_{2n+1},Sx)} \varphi(t)\,dt + \alpha \int_0^{M(x_{2n+1},x)} \varphi(t)\,dt \\
&= \alpha \int_0^{\max\{d(x_{2n+1},x),\, d(x_{2n+1},Tx_{2n+1}),\, d(x,Sx),\, Q_{2n}\}} \varphi(t)\,dt \\
&= \alpha \int_0^{\max\{d(x_{2n+1},x),\, d(x_{2n+1},x_{2n+2}),\, d(x,Sx),\, P_{2n}\}} \varphi(t)\,dt
\end{aligned} \tag{3.7}$$

where $Q_{2n} = \frac{1}{2}[d(x_{2n+1}, Sx)+d(x, Tx_{2n+1})]$ and $P_{2n} = \frac{1}{2}[d(x_{2n+1}, Sx) + d(x, x_{2n+2})]$.

Letting $n \to \infty$ in the above inequality, we conclude that

$$\int_0^{d(x,Sx)} \varphi(t)\,dt \le \alpha \int_0^{H(x,Sx)} \varphi(t)\,dt.$$

So $\int_0^{d(x,Sx)} \varphi(t)\,dt = 0$. Since $Sx \in CB(X)$, we have $x \in Sx$. Similarly, $x \in Tx$. Therefore, T and S have a common fixed point.

Furthermore, if T is single valued, then this common fixed point is unique. In fact, if x and y are two common fixed points for T and S, then

$$\begin{aligned}
&\int_0^{d(x,y)} \varphi(t)\,dt \\
&\le \int_0^{H(\{x\},Sy)} \varphi(t)\,dt = \int_0^{H(Tx,Sy)} \varphi(t)\,dt \\
&\le \alpha \int_0^{M(x,y)} \varphi(t)\,dt \\
&= \alpha \int_0^{\max\left\{d(x,y),\, d(x,Tx),\, d(y,Sy),\, \frac{d(x,Sy)+d(y,Tx)}{2}\right\}} \varphi(t)\,dt \\
&\le \alpha \int_0^{\max\left\{d(x,y),\, 0,\, 0,\, \frac{d(x,y)+d(y,x)}{2}\right\}} \varphi(t)\,dt \\
&= \alpha \int_0^{d(x,y)} \varphi(t)\,dt
\end{aligned} \tag{3.8}$$

Since $0 \le \alpha < 1$, $\int_0^{d(x,y)} \varphi(t)\,dt = 0$, and so $x = y$. □

Remark 3.2. *The last part of the proof of theorem 3.1 shows that if $S, T\colon X \to CB(X)$ are multivalued, x_0 is a common fixed point, and Tx_0 or Sx_0 is a singleton, then the common fixed point of T and S is unique.*

By taking $T = S$ in Theorem 3.1, we get the following corollary that extends the Daffer and Kaneko theorem (Theorem 1.2).

Corollary 3.3. *Let be a complete metric space and $T\colon X \to CB(X)$ be such that*

$$\int_0^{H(Tx,Ty)} \varphi(t)\,dt \le \alpha \int_0^{N(x,y)} \varphi(t)\,dt \tag{20}$$

for some $0 \le \alpha < 1$ and for all $x, y \in X$ (i.e., weak contraction). Then there exists $x_0 \in X$ such that $x_0 \in Tx_0$.

Example 3.4. *Let $X = [0,1]$ be endowed with the Euclidean metric. Let $S, T\colon X \to CB(X)$ be defined by $Tx = [0, x/4]$ and $Sy = \{y/4\}$. Obviously,*

$$\begin{aligned}
&\int_0^{H(Tx,Sy)} \varphi(t)\,dt \\
&= \int_0^{\max\left\{\left|\frac{y}{4}-\frac{x}{4}\right|,\,\frac{y}{4}\right\}} \varphi(t)\,dt \\
&\leq \int_0^{\frac{1}{2}\max\left\{|y-x|,\,\left|y-\frac{y}{4}\right|\right\}} \varphi(t)\,dt \qquad (3.10) \\
&= \int_0^{\frac{1}{2}\max\{d(x,y),\,d(y,Sy)\}} \varphi(t)\,dt \\
&\leq \int_0^{\frac{1}{2}M(x,y)} \varphi(t)\,dt
\end{aligned}$$

So T and S have a common fixed point ($x = 0$), and since S is single valued, this fixed point is unique.

4 Conclusion

We have extended Nadler and Daffer–Kaneko's theorems into Common Fixed Point of Multivalued Generalized φ-Weak Contractive Mappings in Integral type inequality.

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