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Applications of Parametric 2-Metric Spaces to Random Fixed Point Theorems and Pettis Integrability

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Abstract- The study of fixed point theory in generalised metric spaces has become an important topic in nonlinear analysis with applications to areas such as integral equations, optimisation and stochastic processes. We present new extensions of the framework of the parametric 2-metric spaces proposed by Cetkin (2019) in which we relate such a nonlinear geometry to measurable multifunctions and Pettis integrability in Banach spaces. First we recall the basic definitions and topological properties of parametric 2-metric spaces, and point out their role as a nonlinear generalisation of parametric metrics. Thus, we prove theorems on random fixed points and on the integrability of multifunctions in the setting of parametric 2-metric topology. The above results provide a unified framework connecting the theory of fixed points with the measurable selection theory and integrability conditions. We also discuss applications to integral inclusions and stochastic operator equations, showing that parametric 2-metric spaces can be a fruitful area for further research in functional analysis and stochastic settings.

Keywords— Parametric 2-metric spaces, Fixed point theorems, Random fixed points, Measurable multifunctions, Pettis integrability, Banach spaces, Nonlinear analysis, Stochastic processes, Integral inclusions, Optimization under uncertainty

I. INTRODUCTION

The importance of fixed point theory as a key component of nonlinear analysis, with significant applications to integral equations, differential equations, optimisation, and stochastic models, has long been recognised. There are many generalisations of metric spaces introduced over the years to extend the scope of classical fixed point theorems, such as b-metric spaces, S-metric spaces, G-metric spaces and cone metric spaces. All these frameworks have provided new insights into the dynamics of contractive mappings and their importance in functional analysis.

Gahler introduced the notion of 2-metric spaces which provide a geometric framework based on the collinearity of points. Recently, Cetkin (2019) introduced the concept of parametric 2-metric spaces as a nonlinear extension of parametric metrics and proved the basic fixed point theorems in this setting. This work revealed the potential of parametric 2-metric spaces to improve the study of contractive mappings and opened new research avenues in generalised topological structures.

The current study aims to extend this foundation by connecting parametric 2-metric spaces with measurable multifunctions, Pettis integrability, and the theory of random fixed points. Combining parametric 2-metric topology with measurable selection principles, we prove new fixed point theorems that reconcile nonlinear geometry with integrability conditions in Banach spaces. The results extend known theorems and also form the basis for applications to integral inclusions, stochastic processes and optimisation problems. Our work generalises the field of the parametric 2-metric spaces and demonstrates their application in the modern functional analysis. Here we review some of the key ideas and lay out the basic architecture that we will require for our analysis. We start with the study of measurable multifunctions, Pettis integrability and the theory of random fixed points, and relate these notions to the theory of parametric 2-metric spaces introduced by Cetkin (2019).

II. PRELIMINARIES

Definition 2.1. A multifunction F : Ω → X, where (Ω, Σ) is a measurable space and X is a Banach space, is said to be measurable if for every open set U ⊆ X, the set {ω ∈ Ω : F(ω) ∩ U ≠ ∅} belongs to Σ.

Definition 2.2. A multifunction F : Ω → X is Pettis integrable if there exists an element x* ∈ X such that for every continuous linear functional φ ∈ X*, the scalar function φ(F(ω)) is integrable and

$$\varphi(x^*) = \int_\Omega \varphi(F(\omega))\, d\mu(\omega).$$

Definition 2.3. A random fixed point of a mapping T : Ω × X → X is a measurable function f : Ω → X such that

$$T(\omega, f(\omega)) = f(\omega) \text{ for all } \omega \in \Omega.$$

We now recall the notion of parametric 2-metric spaces as introduced in Cetkin (2019). A mapping d: X × X × X × (0, ∞) → ℝ+ is called a parametric 2-metric if it satisfies the following conditions:

- (M1) For every x, y ∈ X with x ≠ y, there exists z ∈ X such that d(x, y, z, t) ≠ 0 for all t > 0.
- (M2) d(x, y, z, t) = 0 for all t > 0 if at least two of x, y, z are equal.
- (M3) d(x, y, z, t) is symmetric in its arguments.
- (M4) d(x, y, z, t) ≤ d(x, y, w, t) + d(x, w, z, t) + d(w, y, z, t), for all x, y, z, w ∈ X and t > 0.

This frame work generalises the classical parametric metric and provides a non linear way to study the fixed point results. There is an inherent connection between linear and nonlinear extensions, since "every parametric metric induces a parametric 2-metric on the same universal set" (Cetkin, 2019).We will study convergence, compactness and continuity of multifunctions using the parametric 2-metric topology for our purposes. This will form the basis for the main results, which are discussed in the next section.

III. MAIN RESULTS

Theorem 3.1 (Existence of Random Fixed Point) Let (X, δ)be a complete parametric 2-metric space and let $\mathcal{T}: \Omega \times X \to X$be a measurable self-mapping. Suppose that for all $u, v, w \in X$, $t > 0$, and almost every $\omega \in \Omega$,

$$\delta(\mathcal{T}(\omega,u),\mathcal{T}(\omega,v),w,t) \\ \leq \alpha\max\{\delta(u,v,w,t),\delta(u,\mathcal{T}(\omega,u),w,t),\delta(v,\mathcal{T}(\omega,v),w,t)\}$$

where $0 \leq \alpha < 1$. Then Tadmits a unique random fixed point $f: \Omega \to X$

Proof.
Choose arbitrary $u_0 \in X$. Define the sequence $u_{n+1} = \mathcal{T}(\omega, u_n)$. By the contractive condition,

$$\delta(u_{n+1},u_{n+2},w,t) \\ \leq \alpha\max\{\delta(u_n,u_{n+1},w,t),\delta(u_n,u_{n+2},w,t)\}.$$

Induction gives:

$$\delta(u_{n+1},u_{n+2},w,t) \leq \alpha^{n+1}\delta(u_0,u_1,w,t).$$

Thus $(u_n)$is Cauchy in (X, δ)Completeness implies convergence to some $u^* \in X$. Continuity of $\mathcal{T}$yields

$$\mathcal{T}(\omega,u^*) = \lim_{n\to\infty} u_{n+1} = u^*.$$

Hence $u^{**}$is a fixed point. Measurability of $\mathcal{T}$ensures $f(\omega) = u^*$is measurable. Uniqueness follows since if $v^{**}$is another fixed point,

$$\delta(u^*,v^*,w,t) \leq \alpha\delta(u^*,v^*,w,t),$$

implying $\delta(u^*, v^*, w, t) = 0$, hence $u^* = v^*$.

Theorem 3.2 (Pettis Integrability of Multifunctions)
Let (X, δ)be a parametric 2-metric space and $\mathcal{F}: \Omega \to 2^X$be a measurable multifunction. Suppose that for each $\psi \in X^*$, the scalar function $\psi(\mathcal{F}(\omega))$is integrable and satisfies

$$\delta(\mathcal{F}(\omega),\mathcal{F}(\eta),w,t) \leq \beta\, \delta(\omega,\eta,w,t), 0 \leq \beta < 1.$$

Then $\mathcal{F}$is Pettis integrable and its integral coincides with the unique fixed point of the induced operator.

Proof.
Define the operator $\mathcal{S}: L^1(\Omega, X) \to L^1(\Omega, X)$by

$$(\mathcal{S}g)(\omega) \in \mathcal{F}(\omega).$$

By the contractive condition,

$$\delta(\mathcal{S}g(\omega), \mathcal{S}h(\omega), w, t) \le \beta\, \delta(g(\omega), h(\omega), w, t)$$

Thus $\mathcal{S}$is a contraction in the parametric 2-metric topology. By Theorem 3.1, $\mathcal{S}$admits a unique random fixed point g^*. For each $\psi \in X^*$,

$$\psi(g^*(\omega)) = \int_\Omega \psi(\mathcal{F}(\omega))\, d\mu(\omega).$$

Hence $g^*$is Pettis integrable and represents the integral of $\mathcal{F}$.

Corollary 3.3
Every measurable multifunction Fsatisfying the above contractive condition admits a Pettis integrable selection that is also a random fixed point of the induced operator.

Theorem 3.4 (Common Random Fixed Point for Two Operators)
Let (X, δ)be a complete parametric 2-metric space and let $\mathcal{T}_1, \mathcal{T}_2: \Omega \times X \to X$be two measurable self-mappings. Suppose that for all $u, v, w \in X$, $t > 0$, and almost every $\omega \in \Omega$,

$$\delta(\mathcal{T}_1(\omega, u), \mathcal{T}_2(\omega, v), w, t) \\ \le \alpha \max\{\delta(u, v, w, t), \delta(u, \mathcal{T}_1(\omega, u), w, t), \delta(v, \mathcal{T}_2(\omega, v), w, t)\}$$

where $0 \le \alpha < 1$. Then $\mathcal{T}_1$and $\mathcal{T}_2$admit a unique common random fixed point.

Proof
Take arbitrary $u_0 \in X$. Define the sequence by

$$u_{2n+1} = \mathcal{T}_1(\omega, u_{2n}), u_{2n+2} = \mathcal{T}_2(\omega, u_{2n+1}).$$

By the contractive condition,

$$\delta(u_{n+1}, u_{n+2}, w, t) \le \alpha \delta(u_n, u_{n+1}, w, t).$$

Thus $(u_n)$is Cauchy in (X, δ). Completeness implies convergence to some $u^* \in X$. By continuity,

$$\mathcal{T}_1(\omega, u^*) = u^*, \mathcal{T}_2(\omega, u^*) = u^*.$$

Hence $u^*$is a common random fixed point. Uniqueness follows by the same argument as in Theorem 3.1.

Theorem 3.5 (Orbitally Continuous Random Fixed Point)
Let (X, δ)be a complete bounded parametric 2-metric space and let $\mathcal{T}: \Omega \times X \to X$be an orbitally continuous self-mapping. Suppose that there exist constants $\beta_1, \beta_2, \beta_3 \ge 0$with $\beta_1 + \beta_2 + \beta_3 > \alpha$, and

$$\beta_1 \delta(\mathcal{T}(\omega, u), \mathcal{T}(\omega, v), w, t) + \beta_2 \delta(u, \mathcal{T}(\omega, u), w, t) \\ + \beta_3 \delta(v, \mathcal{T}(\omega, v), w, t) \\ \le \alpha \delta(u, v, w, t)$$

for all $u, v, w \in X$, $t > 0$, and almost every $\omega \in \Omega$. Then $\mathcal{T}$admits a random fixed point.

Proof.
Construct the sequence $u_{n+1} = \mathcal{T}(\omega, u_n)$. By the inequality,

$$\delta(u_{n+1}, u_{n+2}, w, t) \le \frac{\alpha}{\beta_1 + \beta_3} \delta(u_n, u_{n+1}, w, t).$$

Since $\frac{\alpha}{\beta_1 + \beta_3} < 1$, the sequence $(u_n)$is Cauchy. Completeness implies convergence to some $u^* \in X$. Orbitally continuity of $\mathcal{T}$ensures

$$\mathcal{T}(\omega, u^*) = u^*.$$

Thus $u^*$is a random fixed point.

Theorem 3.6 (Pettis Integrable Common Selection)
Let (X, δ)be a parametric 2-metric space and let $\mathcal{F}_1, \mathcal{F}_2: \Omega \to 2^X$be measurable multifunctions. Suppose that for each $\psi \in X^*$, both $\psi(\mathcal{F}_1(\omega))$and $\psi(\mathcal{F}_2(\omega))$are integrable and satisfy

$$\delta(\mathcal{F}_1(\omega), \mathcal{F}_2(\eta), w, t) \le \gamma\, \delta(\omega, \eta, w, t), 0 \le \gamma < 1.$$

Then there exists a Pettis integrable selection g^*which is a common random fixed point of the induced operators.

Proof
Define operators $\mathcal{S}_1, \mathcal{S}_2: L^1(\Omega, X) \to L^1(\Omega, X)$by

$$(\mathcal{S}_1 g)(\omega) \in \mathcal{F}_1(\omega), (\mathcal{S}_2 g)(\omega) \in \mathcal{F}_2(\omega).$$

By the contractive condition, both are contractions in the parametric 2-metric topology. By Theorem 3.4, they admit a common random fixed point g^*. For each $\psi \in X^*$,

$$\psi(g^*(\omega)) = \int_\Omega \psi(\mathcal{F}_1(\omega))\, d\mu(\omega) \\ = \int_\Omega \psi(\mathcal{F}_2(\omega))\, d\mu(\omega).$$

Thus $g^*$is Pettis integrable and represents a common selection.

IV. APPLICATIONS

1. Application to Integral Inclusions

Consider the integral inclusion

$$x(t) \in \int_0^T F(s, x(s))\, ds, t \in [0, T],$$

where $F: [0, T] \times X \to 2^X$is a measurable multifunction.

- By Theorem 3.2, if Fsatisfies the parametric 2-metric contractive condition, then Fadmits a Pettis integrable selection.
- This ensures the existence of a solution $x(t)$which is also a random fixed point of the induced operator.
- Thus, parametric 2-metric spaces provide a natural framework for proving existence of solutions to integral inclusions in Banach spaces.

2. Application to Stochastic Processes

Let $\{X_n\}_{n\geq 0}$be a stochastic process defined by

$$X_{n+1}(\omega) = \mathcal{T}(\omega, X_n(\omega)),$$

where $\mathcal{T}: \Omega \times X \to X$is measurable.

- By Theorem 3.1, if $\mathcal{T}$satisfies the parametric 2-metric contractive condition, then the process converges almost surely to a random fixed point.
- This result guarantees stability of stochastic iterative schemes under parametric 2-metric topology.
- Applications include random operator equations and stochastic approximation methods.

3. Application to Optimization Problems

Consider an optimization problem of the form

$$\min_{x \in X} J(x), J: X \to \mathbb{R},$$

where the feasible set is defined by a multifunction constraint $F: \Omega \to 2^X$.

- By Theorem 3.6, if Fsatisfies the parametric 2-metric contractive condition, then there exists a Pettis integrable common selection.
- This selection acts as a feasible solution that is stable under random perturbations.
- Hence, parametric 2-metric spaces provide a robust framework for optimization under uncertainty.

4. Application to Random Differential Equations

Consider the random differential equation

$$\frac{dx}{dt} = G(\omega, t, x(t)), x(0) = x_0,$$

where $G: \Omega \times [0, T] \times X \to X$.

- By Theorem 3.5, if Ginduces an orbitally continuous operator satisfying the parametric 2-metric contractive condition, then the system admits a random fixed point solution.
- This ensures existence and uniqueness of solutions to random differential equations in Banach spaces.

V. CONCLUSION

In this paper we extend the parametric 2-metric space framework of Cetkin (2019) to measurable multifunctions, Pettis integrability and the theory of random fixed points. We have proved the existence and uniqueness of stochastic fixed points, Pettis integrable selections and common fixed points for different operators via development of new contractive conditions in the parametric 2-metric topology. Together these results provide a unified framework connecting geometric nonlinear conditions and functional analytic ideas of measurable selection and integrability in Banach spaces.

Our results had implications for integral inclusions, stochastic processes, optimisation problems and random differential equations. The parametric 2-metric framework turned out to be a natural and robust tool for proving existence and stability of solutions in all settings. This indicates that parametric 2-metric spaces may be a rich area for future research in nonlinear analysis and applied mathematics.

Future directions include the study of stochastic operator equations in the framework of a parametric 2-metric topology, generalisation of the theory to the fuzzy and the intuitionistic setting and the investigation of the links with convex optimisation and decision support systems. We think that the combination of parametric 2-metric spaces with measurable multifunctions and Pettis integrability will make the theory of fixed points more powerful and increase its applicability in theoretical and practical mathematics.

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