

Equilibrium Structure and Local Motion in the Photogravitational Restricted Four-Body Problem with a Circumstellar Belt

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Abstract- The present study investigates the equilibrium structure and local stability characteristics of a planar photo gravitational restricted four-body problem under the combined influence of radiation pressure and an axisymmetric circumstellar belt. The dynamical model considers three primaries arranged in an equilateral triangular configuration, while the motion of an infinitesimal particle is governed by radiation-modified gravitational forces and the gravitational potential of the belt. The equilibrium equations are solved numerically using a systematic search procedure with multiple initial conditions, followed by residual verification and eigenvalue-based stability analysis. The influence of the dominant-primary radiation factor and belt mass parameter on equilibrium locations is examined. The results show that radiation pressure produces systematic displacement of equilibrium points by modifying the effective gravitational balance, although the stability classification remains unchanged for the investigated parameter sets. In the stellar application, the numerical search identifies additional axial equilibrium solutions beyond the previously reported configurations. These solutions are verified using high-precision calculations to confirm their satisfaction of the equilibrium conditions. Linear stability analysis reveals that the near-origin equilibrium possesses purely imaginary characteristic roots and is therefore linearly stable within the adopted infinitesimal-particle approximation, whereas the remaining detected equilibria exhibit instability due to characteristic roots with positive real components. The results demonstrate that circumstellar belt perturbations and radiation effects significantly influence equilibrium multiplicity and local dynamics in modified restricted four-body systems. The study highlights the importance of accurate numerical exploration and stability verification for identifying closely located equilibrium solutions. Further investigations involving nonlinear stability analysis and global dynamical evolution are required to extend the present results beyond the linear approximation.

Keywords— Restricted four-body problem; photo gravitational dynamics; circumstellar belt; equilibrium points; radiation pressure; linear stability.

I. INTRODUCTION

The restricted four-body problem (R4BP) is an extension of the classical restricted three-body problem in which an infinitesimal particle moves under the gravitational influence of three finite masses. Among different configurations, the planar equilateral arrangement of the primaries has attracted considerable attention because it provides

a fundamental model for studying equilibrium structures, periodic motion, and stability behaviour in multi-body gravitational systems. The existence and characteristics of equilibrium points in this

configuration depend strongly on the mass distribution and dynamical parameters of the system (Alvarez-Ramirez & Vidal, 2009; Baltagiannis & Papadakis, 2011).

In astrophysical environments, additional physical effects can significantly modify the classical restricted four-body framework. Radiation pressure is one of the most important perturbations in systems containing luminous bodies because it reduces the effective gravitational attraction experienced by small particles. The inclusion of radiation parameters changes the equilibrium locations, modifies the effective potential, and influences the local stability properties of libration points (Kushvah, 2009; Singh & Vincent, 2016). Recent studies have further demonstrated that radiation pressure combined with other perturbations can produce significant changes in equilibrium configurations and orbital behaviour in restricted few-body systems (Osorio-Vargas et al., 2020; Pal et al., 2024).

Distributed mass structures, such as circumstellar and circumbinary belts, provide another physically relevant perturbation in restricted dynamical models. These structures introduce an additional gravitational potential that can alter the equilibrium topology, create new equilibrium solutions, and modify the stability characteristics of existing points. Belt effects have been incorporated into generalized Chermnykh-like and photogravitational problems, where variations in disc parameters were shown to influence resonance conditions, equilibrium positions, and stability regions (Jiang & Yeh, 2006; Kishor & Kushvah, 2013). Similarly, studies involving radiating primaries surrounded by belts have demonstrated that the additional gravitational contribution of distributed matter plays an important role in determining the dynamical structure of the system (Singh & Haruna, 2020; Singh & Isah, 2021).

The accurate determination of equilibrium points is a fundamental step in understanding restricted multi-body dynamics. However, numerical identification of equilibria requires careful verification because closely located solutions, especially near the origin or along symmetry axes, may be overlooked depending on the initial search strategy. Previous investigations have highlighted the importance of combining numerical approaches

with stability analysis and mathematical verification when analysing equilibrium multiplicity and local dynamics (Papadouris & Papadakis, 2013; Ramírez & Álvarez-Ramírez, 2022).

The present study investigates the planar photogravitational restricted four-body problem with an additional axisymmetric circumstellar belt potential. The three primaries are assumed to form an equilateral triangular configuration, while the infinitesimal particle moves within the resulting rotating effective potential. The objectives of this work are to determine and verify equilibrium solutions, analyse their displacement under variations of radiation pressure and belt mass, evaluate their linear stability through characteristic roots, and examine the influence of the belt contribution on equilibrium multiplicity. Numerical verification is performed using multiple initial conditions, residual analysis, and independent eigenvalue calculations to ensure the reliability of the obtained solutions.

II. LITERATURE REVIEW

The restricted four-body problem has been investigated extensively as a natural extension of the restricted three-body problem for describing gravitational interactions involving three primary bodies and a massless particle. Early studies established the dependence of equilibrium configurations on the masses and spatial arrangement of the primaries. Alvarez-Ramirez and Vidal (2009) analysed dynamical properties of the restricted four-body problem and demonstrated that equilibrium structures are strongly influenced by system parameters. Baltagiannis and Papadakis (2011) further examined equilibrium points and their stability, providing important results on the existence and classification of stationary solutions. The equilateral restricted four-body configuration has continued to receive attention because of its relevance to periodic motion and stability analysis. Ramírez and Álvarez-Ramírez (2022) reviewed equilibrium points and linear stability in the planar equilateral restricted four-body problem, showing that equilibrium characteristics depend on the mass parameter and dynamical assumptions. Bakker and

Freeman (2023) extended the analysis of relative equilibria and periodic solutions in circular planar multi-body configurations, demonstrating the complex relationship between equilibrium families and orbital behaviour.

The introduction of radiation pressure has significantly expanded the classical restricted-body models. Radiation pressure modifies the gravitational attraction of luminous primaries by introducing radiation-reduction factors into the equations of motion. Kushvah (2009) investigated the influence of the Poynting–Robertson effect on the linear stability of equilibrium points in a generalized photogravitational Chermnykh problem and showed that radiation-related perturbations can alter stability conditions. Singh and Vincent (2016) studied equilibrium points in the restricted four-body problem with radiation pressure and demonstrated that radiative effects influence both equilibrium positions and their stability characteristics. Osorio-Vargas et al. (2020) further investigated orbital dynamics in the photogravitational restricted four-body problem and showed that radiation parameters affect the motion around equilibrium regions.

The presence of circumstellar and circumbinary material introduces additional gravitational effects that cannot be neglected in realistic astrophysical systems. Jiang and Yeh (2006) studied Chermnykh-like problems involving additional perturbing effects and provided a framework for analysing modified gravitational potentials. Kishor and Kushvah (2013) investigated generalized photogravitational Chermnykh-like problems with a disc and demonstrated that disc parameters influence equilibrium stability and resonant behaviour.

Belt and disc perturbations have also been incorporated into systems containing radiating or non-spherical primaries. Singh and Haruna (2020) examined triangular point dynamics in systems containing radiating heterogeneous primaries surrounded by a belt and showed that the belt contribution modifies the equilibrium structure. Singh and Isah (2021) analysed collinear libration points under combined radiation, triaxiality, and belt

effects, confirming that additional gravitational potentials significantly influence stationary solutions. Vincent et al. (2022) investigated equilibrium existence and stability under the influence of circumbinary disc effects and demonstrated the importance of distributed mass contributions in modified restricted systems. Umar et al. (2021) similarly studied circumbinary disc perturbations in a photogravitational elliptic restricted three-body problem and reported changes in equilibrium behaviour due to disc-induced gravitational effects. The mathematical representation of distributed mass systems often employs analytical potentials capable of describing flattened stellar structures. The Miyamoto–Nagai potential has been widely used as a convenient representation of axisymmetric disc-like structures because it provides a smooth gravitational contribution while maintaining analytical tractability (Miyamoto & Nagai, 1975). Such potentials are suitable for incorporating circumstellar belt effects into restricted dynamical models.

Despite extensive research on radiation pressure and belt perturbations individually, the combined influence of radiation-modified gravitational forces and an axisymmetric circumstellar belt on equilibrium multiplicity and local stability in the planar photogravitational restricted four-body problem remains insufficiently explored. Therefore, this study focuses on the numerical determination, verification, and stability analysis of equilibrium points under the simultaneous effects of radiation pressure and circumstellar belt gravity.

III. MATHEMATICAL FORMULATION AND EQUATIONS OF MOTION

1. Dynamical Model and Coordinate System

Consider an infinitesimal particle moving under the gravitational attraction of three radiating primaries arranged in an equilateral triangular configuration. The primaries execute prescribed circular motion about their common centre of mass, while the particle's negligible mass leaves their motion unaffected. The model follows the equilateral photogravitational restricted four-body

formulations of Papadouris and Papadakis (2013) and Singh and Vincent (2016).

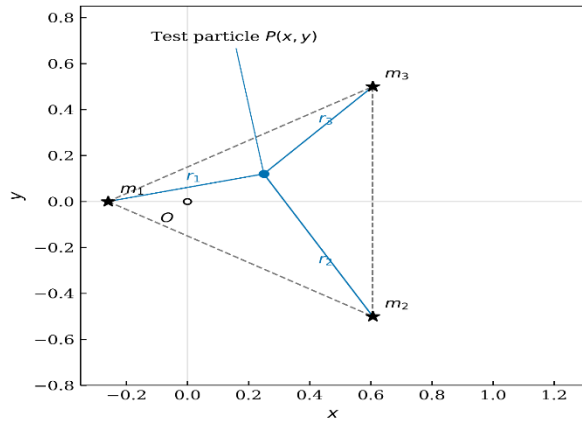
The motion is described in a planar synodic frame rotating with constant angular velocity n . Dimensionless units are adopted such that the gravitational constant, total primary mass, and mutual separation of the primaries are unity. The two smaller primaries have equal masses and radiation factors:

$$m_1 = 1 - 2\mu, m_2 = m_3 = \mu, q_2 = q_3, 0 < \mu < \frac{1}{3}. \quad (1)$$

With the origin at the centre of mass, the primary coordinates are

$$\begin{aligned} (x_1, y_1) &= (-\sqrt{3}\mu, 0), \\ (x_2, y_2) &= \left(\frac{\sqrt{3}}{2}, (1 - 2\mu) - \frac{1}{2}\right), \\ (x_3, y_3) &= \left(\frac{\sqrt{3}}{2}, (1 - 2\mu)\frac{1}{2}\right). \end{aligned} \quad (2)$$

Radiation pressure is represented by constant reduction factors. Unity corresponds to the absence of radiation pressure, whereas smaller values



indicate a greater reduction in effective gravitational attraction. The distance between the particle and each primary is

$$r_i = \sqrt{(x - x_i)^2 + (y - y_i)^2}, 0 < q_i \leq 1, i = 1, 2, 3. \quad (3)$$

An axisymmetric belt centred at the origin provides an additional gravitational contribution. Velocity-dependent radiation drag and other dissipative forces are excluded.

Figure 1. Configuration of the three primaries and the infinitesimal particle in the rotating frame.

2. Effective Potential and Equations of Motion

The belt contribution is represented by the planar reduction of the Miyamoto–Nagai potential (Miyamoto and Nagai, 1975):

$$\Omega_b(x, y) = \frac{M_b}{\sqrt{x^2 + y^2 + T^2}}, T = a + b > 0, \quad (4)$$

where M_b is the dimensionless belt mass and a and b are structural scale parameters. The physical gravitational potential is negative; therefore, the positive contribution in Equation (4) produces an attractive force towards the origin.

Following the prescription adopted by Vincent et al. (2023), the angular velocity is

$$n^2 = 1 + \frac{2M_b r_c}{(r_c^2 + T^2)^{3/2}}, r_c = \sqrt{1 - \mu + \mu^2}. \quad (5)$$

The reference quantity r_c remains fixed for each mass parameter. This angular velocity is prescribed within the restricted model rather than obtained from a separate solution for the belt-modified motion of the primaries. The primary configuration is consequently treated as imposed, and the stability analysis concerns the infinitesimal particle.

The effective potential is

$$\Omega(x, y) = \frac{n^2}{2}(x^2 + y^2) + \sum_{i=1}^3 \frac{m_i q_i}{r_i} + \frac{M_b}{\sqrt{x^2 + y^2 + T^2}}. \quad (6)$$

The equations of motion in the rotating frame are

$$\ddot{x} - 2n\dot{y} = \Omega_x, \ddot{y} + 2n\dot{x} = \Omega_y, \quad (7)$$

where overdots denote differentiation with respect to dimensionless time and subscripts denote partial derivatives. The force components are

$$\Omega_x = n^2 x - \sum_{i=1}^3 \frac{m_i q_i (x - x_i)}{r_i^3} - \frac{M_b x}{(x^2 + y^2 + T^2)^{3/2}}, \quad (8)$$

$$\Omega_y = n^2 y - \sum_{i=1}^3 \frac{m_i q_i (y - y_i)}{r_i^3} - \frac{M_b y}{(x^2 + y^2 + T^2)^{3/2}}. \quad (9)$$

The belt-free limit recovers unit angular velocity and the corresponding photogravitational system. The primary locations are singularities of the potential and are excluded from the particle domain.

3. Existence and Position of Equilibrium Points

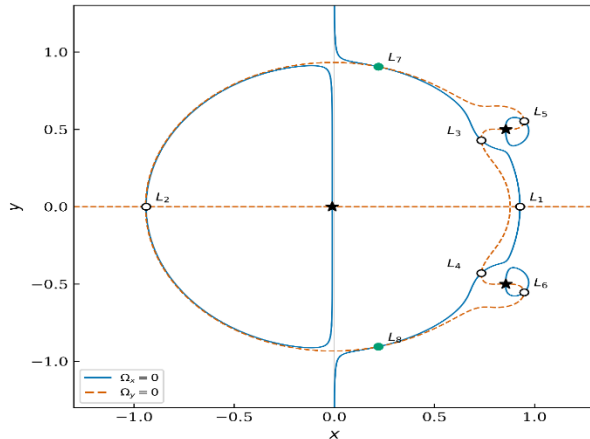
An equilibrium point is stationary relative to the rotating frame. Its velocity and acceleration vanish, giving

$$\Omega_x(x_0, y_0) = 0, \Omega_y(x_0, y_0) = 0. \quad (10)$$

The equal masses and radiation factors of the smaller primaries imply reflection symmetry:

$$\Omega(x, -y) = \Omega(x, y). \quad (11)$$

The equilibria are therefore classified as axial or non-collinear. Axial points lie on the horizontal axis, where the vertical force vanishes identically. Defining



$$X = x - \frac{\sqrt{3}}{2}(1 - 2\mu),$$

their coordinates satisfy

$$n^2 x - \frac{(1 - 2\mu)q_1(x + \sqrt{3}\mu)}{|x + \sqrt{3}\mu|^3} - \frac{2\mu q_2 X}{(X^2 + \frac{1}{4})^{3/2}} - \frac{M_b x}{(x^2 + T^2)^{3/2}} = 0. \quad (12)$$

Non-collinear equilibria require simultaneous solution of Equation (10) and occur in reflected pairs. Their number depends on the prescribed parameters and is not fixed in advance. Equilibrium labels are maintained across parameter variations, with additional axial solutions assigned distinct labels.

Figure 2. Equilibrium locations at intersections of the horizontal and vertical force-null curves.

4. Linear Stability Analysis

The local stability of an equilibrium is examined by introducing small perturbations:

$$x = x_0 + \xi, y = y_0 + \eta. \quad (13)$$

Expansion of the force components about the equilibrium, retaining first-order terms, gives

$$\ddot{\xi} - 2n\dot{\eta} = \Omega_{xx}^0 \xi + \Omega_{xy}^0 \eta, \quad (14)$$

$$\ddot{\eta} + 2n\dot{\xi} = \Omega_{xy}^0 \xi + \Omega_{yy}^0 \eta, \quad (15)$$

where the superscript zero denotes evaluation at the equilibrium. Substituting

$$(\xi, \eta) = (A, B)e^{\lambda t} \quad (16)$$

yields the characteristic equation

$$\lambda^4 + (4n^2 - \Omega_{xx}^0 - \Omega_{yy}^0)\lambda^2 + \Omega_{xx}^0 \Omega_{yy}^0 - (\Omega_{xy}^0)^2 = 0. \quad (17)$$

Introducing

$$R = \sqrt{x^2 + y^2 + T^2}, d_{xi} = x - x_i, d_{yi} = y - y_i,$$

the second derivatives are

$$\Omega_{xx} = n^2 + \sum_{i=1}^3 m_i q_i \left(\frac{3d_{xi}^2}{r_i^5} - \frac{1}{r_i^3} \right) + M_b \left(\frac{3x^2}{R^5} - \frac{1}{R^3} \right), \quad (18)$$

$$\Omega_{yy} = n^2 + \sum_{i=1}^3 m_i q_i \left(\frac{3d_{yi}^2}{r_i^5} - \frac{1}{r_i^3} \right) + M_b \left(\frac{3y^2}{R^5} - \frac{1}{R^3} \right), \quad (19)$$

$$\Omega_{xy} = 3 \sum_{i=1}^3 \frac{m_i q_i d_{xi} d_{yi}}{r_i^5} + \frac{3M_b xy}{R^5}. \quad (20)$$

Define the characteristic coefficients as

$$\mathcal{B} = 4n^2 - \Omega_{xx}^0 - \Omega_{yy}^0, \mathcal{D} = \Omega_{xx}^0 \Omega_{yy}^0 - (\Omega_{xy}^0)^2.$$

Distinct, purely imaginary characteristic roots occur when

$$\mathcal{B} > 0, \mathcal{D} > 0, \mathcal{B}^2 - 4\mathcal{D} > 0. \quad (21)$$

These conditions yield bounded oscillations in the linearised system. A root with a positive real part establishes linear instability. Repeated imaginary roots and zero roots require separate examination, and linear stability alone does not establish nonlinear stability.

At axial equilibria, symmetry makes the mixed derivative vanish. A negative product of the diagonal derivatives then implies instability. Reflected non-

collinear points have identical characteristic polynomials and stability classifications.

5. Numerical Formulation and Parameter Studies

The equilibrium equations form a nonlinear system,

$$F(z) = \begin{pmatrix} \Omega_x \\ \Omega_y \end{pmatrix} = 0, \quad z = \begin{pmatrix} x \\ y \end{pmatrix}. \quad (22)$$

Its analytical Jacobian is the Hessian of the effective potential:

$$J = \begin{pmatrix} \Omega_{xx} & \Omega_{xy} \\ \Omega_{xy} & \Omega_{yy} \end{pmatrix}. \quad (23)$$

This formulation permits iterative root determination from multiple initial positions. Candidate equilibria are assessed through their force residuals, with primary singularities excluded and duplicate roots distinguished from separate solutions. Search refinement is necessary to assess the detected equilibrium count.

The comparative parameter studies vary the radiation factor of the dominant primary, the common radiation factor of the smaller primaries, and the belt mass separately. The dominant-primary radiation study adopts

$$\mu = 0.006, \quad q_2 = 0.9, \quad M_b = 0.05, \quad T = 0.02,$$

while the common-radiation study adopts

$$\mu = 0.006, \quad q_1 = 0.9, \quad M_b = 0.05, \quad T = 0.02.$$

The belt-mass interval is

$$0 \leq M_b \leq 0.12.$$

The remaining parameters are fixed within each comparison. Angular velocity is updated whenever the belt mass changes, so the belt response incorporates both its direct gravitational contribution and the associated change in the centrifugal term. A decrease in a radiation factor corresponds to stronger radiation pressure.

For each equilibrium, the characteristic roots follow from Equation (17). They can also be obtained from the first-order linearisation matrix,

$$A_0 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \Omega_{xx}^0 & \Omega_{xy}^0 & 0 & 2n \\ \Omega_{xy}^0 & \Omega_{yy}^0 & -2n & 0 \end{pmatrix}, \quad (24)$$

providing an independent consistency check on the characteristic equation.

6. Stellar Application

The idealised Ross 104 – Ross 775a – Ross 775b application adopts the parameters reported by Vincent et al. (2023):

$$\mu = 0.261363636, \quad q_1 = 0.9995264, \quad q_2 = q_3 = 0.9913514, \quad T = 0.02. \quad (25)$$

These values are applied within the prescribed equilateral configuration. The application concerns particle dynamics in this idealised model rather than an independent reconstruction of the observed stellar orbits.

Vincent et al. (2023) report eight equilibrium points for the smaller belt mass and ten for the larger belt mass:

$$M_b = 0.01: \text{ eight reported equilibria,}$$

$$M_b = 0.05: \text{ ten reported equilibria.}$$

These published cases provide benchmarks for assessing equilibrium multiplicity and local stability. Their reported coordinates and characteristic roots remain distinct from independently reproduced numerical results.

7. Jacobi Integral and Zero-Velocity Geometry

Because the effective potential is time independent, the equations of motion admit the Jacobi integral

$$C = 2\Omega(x, y) - (\dot{x}^2 + \dot{y}^2), \quad (26)$$

where C remains constant along a trajectory. The zero-velocity condition gives

$$C = 2\Omega(x, y). \quad (27)$$

Motion is energetically accessible where

$$2\Omega(x, y) \geq C,$$

and forbidden where

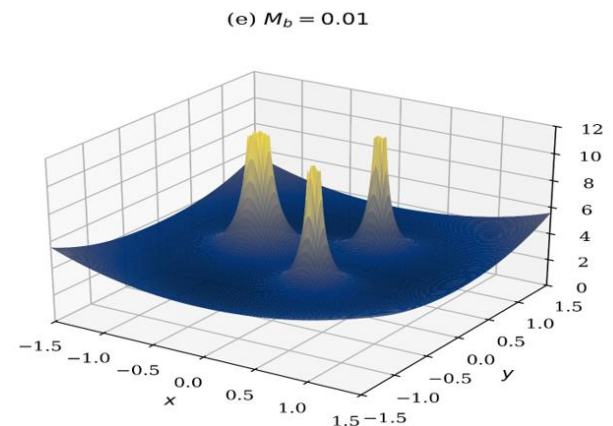
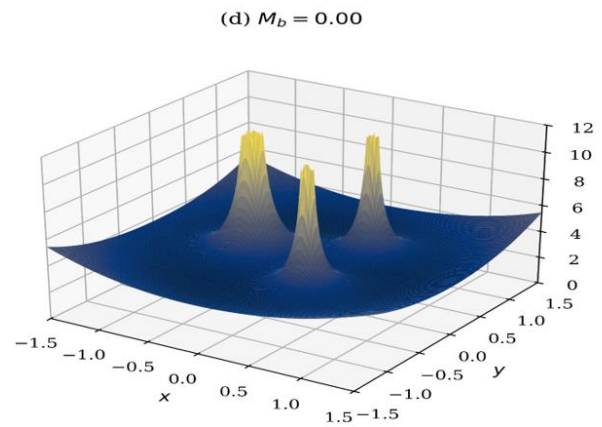
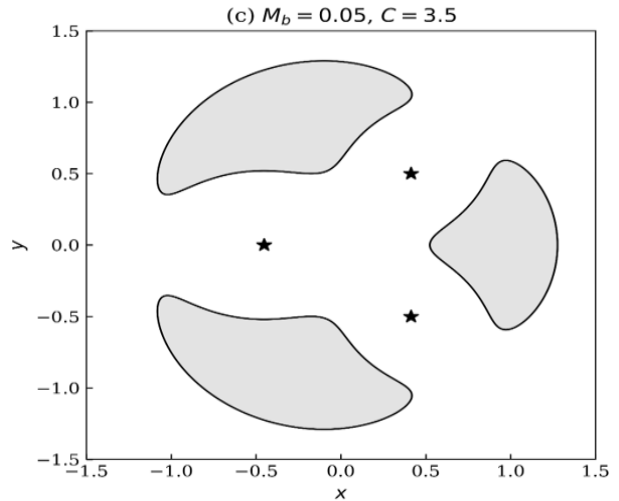
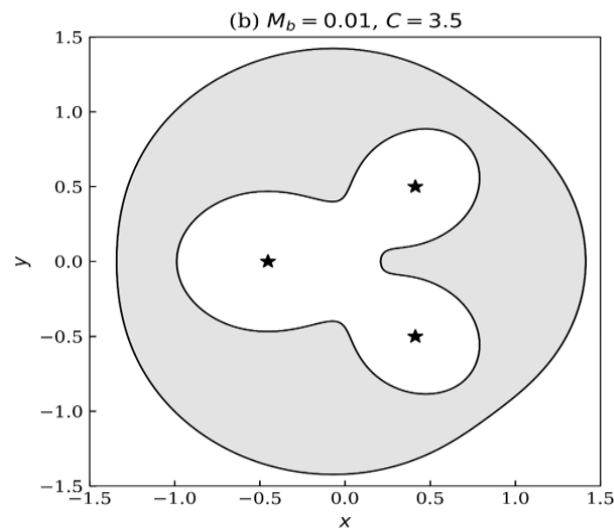
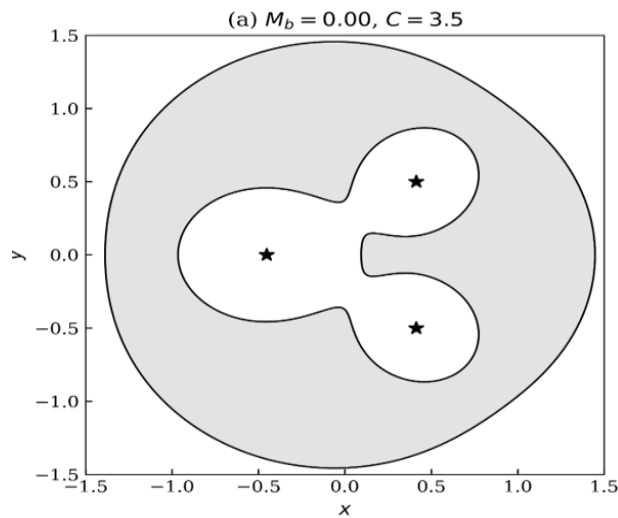
$$2\Omega(x, y) < C.$$

Zero-velocity curves are contours of Equation (27), while the corresponding surface represents twice the

effective potential over the configuration plane. Each equilibrium has a critical Jacobi value,

$$C_j = 2\Omega(x_j, y_j). \quad (28)$$

At nondegenerate saddle points, these values identify possible changes in the connectivity of accessible regions. Comparisons at a common Jacobi constant reveal how radiation and belt gravity alter energetic accessibility. Where different critical constants are used, their values are stated explicitly. The zero-velocity geometry defines energetic boundaries but does not, by itself, establish chaotic motion, escape, or long-term confinement.



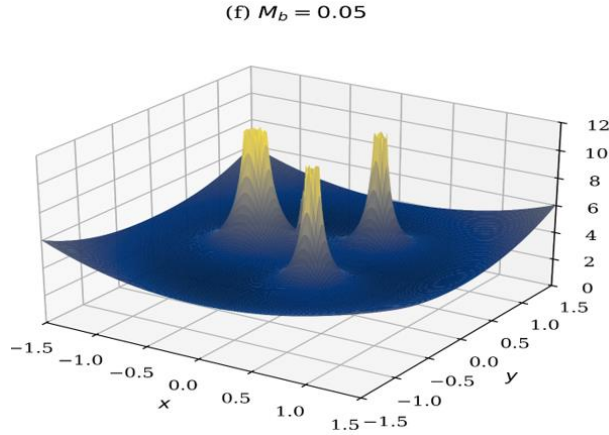


Figure 3. Zero-velocity curves and surfaces for selected belt masses and Jacobi constants

IV. RESULTS AND DISCUSSION

1. Numerical Accuracy

Eight equilibria were detected in each sampled dominant-primary radiation case, whereas ten were detected in each stellar belt case. The force residuals at the unrounded equilibrium coordinates satisfied:

$$\|F(x_0, y_0)\|_{\infty} < 1.3 \times 10^{-14}$$

The characteristic roots obtained from Equation (17) agreed with the eigenvalues of the first-order linearisation matrix within:

$$\max_j \min_k |\lambda_j^{\text{matrix}} - \lambda_k^{\text{polynomial}}| < 6 \times 10^{-13}$$

The additional axial roots in the stellar application were independently verified using 60-digit decimal precision arithmetic. These checks confirm the numerical accuracy of the detected solutions and their local stability classifications. However, the numerical search alone does not establish the completeness of the equilibrium set.

2. Influence of the Dominant-Primary Radiation Factor

The dominant-primary radiation factor was varied while maintaining the remaining parameters fixed at:

$$\mu = 0.006, \quad q_2 = q_3 = 0.9, \quad M_b = 0.05, \quad T = 0.02$$

Eight equilibria were detected at each sampled radiation factor: two axial points and three reflected non-collinear pairs.

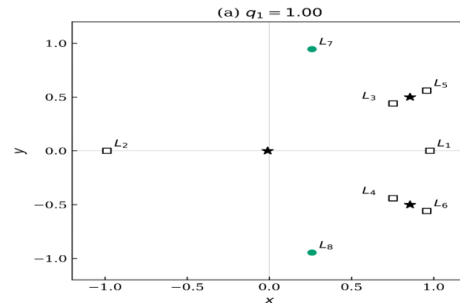
Table 1. Equilibrium coordinates for variation of the dominant-primary radiation factor.

Equilibrium	$q_1 = 1.00$	$q_1 = 0.85$	$q_1 = 0.70$
L_1	(0.977043,0)	(0.926891,0)	(0.870646,0)
L_2	(-0.988149,0)	(-0.939046,0)	(-0.884127,0)
$L_{3,4}$	(0.752608, ± 0.440456)	(0.733892, ± 0.429518)	(0.707169, ± 0.413699)
$L_{5,6}$	(0.957651, ± 0.559552)	(0.947551, ± 0.553516)	(0.939267, ± 0.548605)
$L_{7,8}$	(0.259063, ± 0.945657)	(0.220535, ± 0.904982)	(0.181070, ± 0.857736)

Decreasing the radiation factor strengthens radiation pressure, thereby reducing the effective attraction of the dominant primary. Across the sampled values, both axial equilibria shift towards the origin. The non-collinear pairs exhibit decreasing horizontal coordinates and vertical magnitudes, with the largest displacement occurring for $L_{7,8}$.

The calculated coordinates reproduce the corresponding radiation cases tabulated by Vincent et al. (2023) to the reported precision, serving as a numerical benchmark for the adopted formulation. The observed trends apply specifically to the

evaluated parameter sets and do not guarantee monotonic behavior over the full radiation interval.



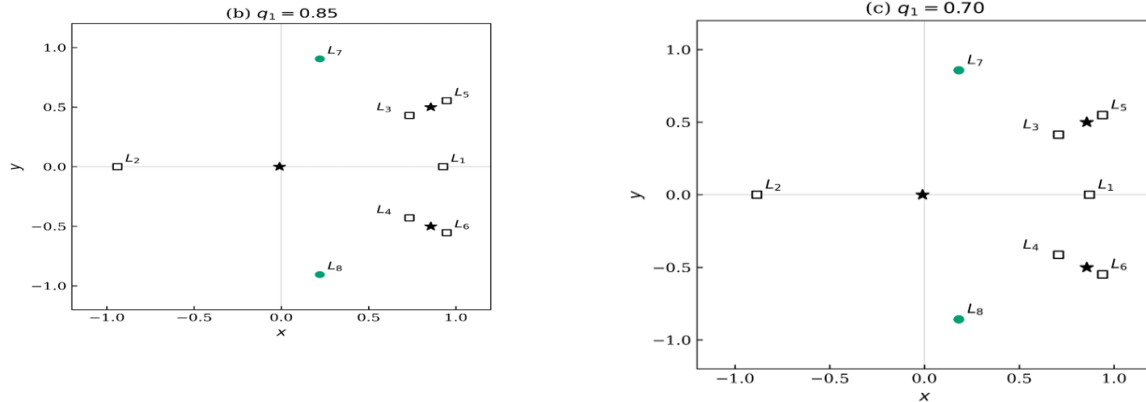


Figure 4. Equilibrium displacement with variation of the dominant-primary radiation factor.

3. Linear Stability in the Radiation Study

The stability classifications remain invariant across the three sampled radiation factors. Both axial equilibria and the pairs $L_{3,4}$ and $L_{5,6}$ possess characteristic roots with positive real parts and are linearly unstable. The pair $L_{7,8}$ possesses distinct, purely imaginary roots and is linearly stable.

Table 2. Characteristic roots for the intermediate radiation case ($q_1 = 0.85$).

Equilibrium	First Characteristic Pair	Second Characteristic Pair	Classification
L_1	$-0.347271 \pm 0.798708i$	$0.347271 \pm 0.798708i$	Unstable
L_2	± 0.169843	$\pm 1.058010i$	Unstable
$L_{3,4}$	± 2.187395	$\pm 1.906525i$	Unstable
$L_{5,6}$	± 2.901314	$\pm 2.338461i$	Unstable
$L_{7,8}$	$\pm 0.998373i$	$\pm 0.312582i$	Linearly stable

For L_1 , the complex quartet contains roots with positive real parts. The remaining unstable equilibria possess a real pair containing a positive root. Radiation variation alters equilibrium positions and characteristic frequencies without inducing a stability transition in the sampled cases.

The purely imaginary roots of $L_{7,8}$ indicate bounded perturbations in the linearized system; however, they do not establish nonlinear stability or long-term spatial confinement.

4. Equilibrium Multiplicity in the Stellar Application

The idealized Ross 104 – Ross 775a – Ross 775b application uses the parameters adopted from Vincent et al. (2023):

$$\mu = 0.261363636, \quad q_1 = 0.9995264, \quad q_2 = q_3 = 0.9913514, \quad T = 0.02$$

Ten equilibria were detected at both belt masses, comprising four axial points and three reflected non-collinear pairs.

Table 3. Equilibrium coordinates for the idealized Ross stellar configuration.

Equilibrium	Mb=0.01	Mb=0.05
L_1	(0.943018,0)	(0.922616,0)
L_2	(-1.148375,0)	(-1.129201,0)
L_{n1}	(-0.001254,0)	(-0.000247,0)

L_{n2}	$(-0.059627, 0)$	$(-0.115165, 0)$
$L_{3,4}$	$(0.072860, \pm 0.238108)$	$(0.136703, \pm 0.222941)$
$L_{5,6}$	$(0.761376, \pm 0.909749)$	$(0.749644, \pm 0.896645)$
$L_{7,8}$	$(-0.296565, \pm 0.871820)$	$(-0.289553, \pm 0.856112)$

The calculated coordinates closely reproduce the published entries, with minor discrepancies restricted to the final reported digits of certain axial coordinates. At the smaller belt mass ($M_b = 0.01$), two additional axial solutions were detected beyond the eight equilibria listed in the published study:

$$x_{n1} = -0.00125377331172758, \quad x_{n2} = -0.05962695866314026$$

Both solutions satisfy the reduced axial equilibrium equation under high-precision calculations. The vertical force component vanishes identically due to reflection symmetry, proving that the stated system admits at least ten equilibria at the smaller belt mass. This finding contrasts with the eight-point count reported by Vincent et al. (2023). The discrepancy reflects a difference in equilibrium detection rather than a modification of the model equations. Identifying its root cause would require auditing the original computational implementation.

Increasing the belt mass draws L_{n1} closer to the origin, whereas L_{n2} shifts further along the negative horizontal axis. The remaining equilibria display varied displacement trends driven jointly by the direct gravitational attraction of the belt and the corresponding adjustment in angular velocity.

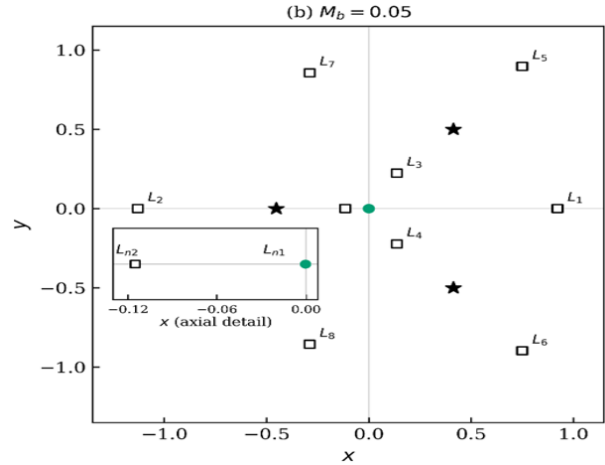
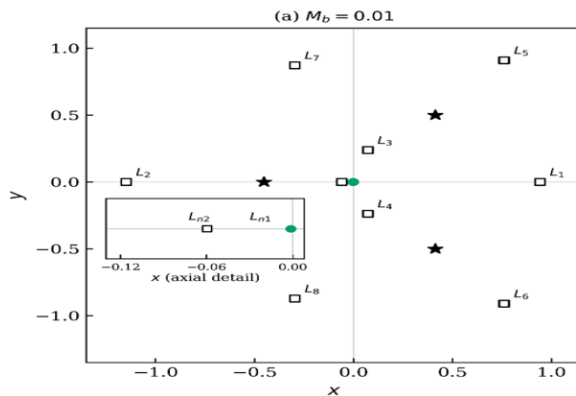


Figure 5. Detected equilibria of the idealized Ross stellar configuration at the two belt masses.

5. Stability of the Stellar Equilibria

The near-origin equilibrium L_{n1} yields distinct, purely imaginary characteristic roots for both belt masses:

$$M_b = 0.01: \quad \lambda = \pm 34.063537i, \quad \pm 36.130189i$$

$$M_b = 0.05: \quad \lambda = \pm 77.954965i, \quad \pm 80.077993i$$

Consequently, L_{n1} is linearly stable in both evaluated configurations. The larger belt mass leads to higher characteristic frequencies, translating to faster small-amplitude oscillations in dimensionless time.

The second additional axial point, L_{n2} , features a real characteristic pair:

$$M_b = 0.01: \quad \lambda = \pm 9.108533, \quad \pm 6.896864i$$

$$M_b = 0.05: \quad \lambda = \pm 9.142998, \quad \pm 6.597923i$$

The presence of a positive real root confirms linear instability at both belt masses. Each of the remaining eight stellar equilibria likewise exhibits characteristic roots with positive real parts. Among all ten detected points, only L_{n1} is linearly stable in the evaluated cases.

Identifying this stable equilibrium alters the dynamical picture of the smaller-belt configuration. Under the adopted model, the equilibrium set contains a linearly stable near-origin point. Note that this stability applies strictly to an infinitesimal test

particle and does not imply the stability of the primary stellar configuration itself.

V. DISCUSSION

The present results demonstrate that radiation pressure and circumstellar belt gravity play important roles in determining the equilibrium topology of the photogravitational restricted four-body problem. The numerical analysis shows that variations in the radiation factor modify the effective gravitational environment and lead to measurable shifts in equilibrium positions. These changes arise from the altered balance between radiation-reduced gravitational attraction, centrifugal effects, and mutual gravitational interactions among the primaries. Similar parameter-dependent displacement behaviour has been reported in other modified restricted-body models involving radiation pressure and additional perturbing forces.

The stability investigation indicates that changes in equilibrium location do not necessarily result in changes in local stability characteristics. For the examined radiation-factor values, the same stability classification is preserved, with most equilibria remaining linearly unstable and the upper and lower off-axis pair maintaining purely imaginary characteristic roots. This confirms that positional sensitivity and stability transition are distinct dynamical effects and must be analysed independently.

A significant finding of this study is the detection of additional axial equilibrium solutions in the idealised stellar configuration. These solutions were obtained through an extensive numerical search and subsequently verified using high-precision calculations. The result demonstrates that equilibrium structures in perturbed restricted-body systems may contain closely spaced solutions that can remain undetected when conventional initial guesses or limited search regions are employed. Accurate equilibrium identification is therefore essential before interpreting the dynamical properties of such systems.

The circumstellar belt introduces an additional gravitational contribution that modifies the effective potential and influences the distribution of equilibrium points. Increasing the belt mass produces systematic changes in equilibrium locations, although the response varies among different equilibrium branches. This behaviour reflects the complex interaction between distributed mass perturbations and the rotating-frame dynamics. Similar effects have been observed in restricted-body models incorporating discs, belts, and other extended mass distributions.

The linear stability results show that the near-origin equilibrium identified in the stellar application is linearly stable for the investigated belt masses, while the remaining equilibria exhibit unstable characteristics. However, this conclusion is restricted to the linearised motion of an infinitesimal particle near the equilibrium positions. Linear stability does not guarantee nonlinear stability or long-term bounded motion, and therefore further investigations involving nonlinear analysis, periodic orbit continuation, and long-term numerical integration are required.

Overall, the study extends the understanding of equilibrium multiplicity and local dynamics in photogravitational restricted four-body systems by demonstrating the combined effects of radiation pressure and circumstellar belt perturbations. The results emphasise the importance of rigorous numerical verification when analysing complex gravitational configurations with multiple perturbing mechanisms.

VI. CONCLUSION

This study has examined the equilibrium structure and linear stability behaviour of a planar photogravitational restricted four-body problem influenced by an axisymmetric circumstellar belt. A mathematical model incorporating radiation-modified gravitational forces and belt-induced perturbation has been developed, and the resulting equilibrium configurations have been investigated through numerical solution and eigenvalue analysis.

The results show that radiation pressure modifies the spatial distribution of equilibrium points by shifting their positions, while maintaining consistent stability classifications within the investigated parameter range. The inclusion of the circumstellar belt further alters the equilibrium topology and leads to the detection of additional axial solutions in the stellar application. These solutions were verified through high-precision calculations, confirming their validity within the adopted dynamical model.

The linear stability analysis indicates that only the near-origin equilibrium exhibits linearly stable behaviour through purely imaginary characteristic roots, whereas the remaining equilibria are unstable due to the presence of eigenvalues with positive real components. These findings demonstrate that additional gravitational perturbations can significantly influence equilibrium multiplicity and local dynamical behaviour.

The present investigation is limited to the planar restricted model, prescribed parameter values, and linear stability framework. Future studies should consider nonlinear stability, bifurcation analysis, and long-term orbital evolution to obtain a more comprehensive understanding of particle dynamics in realistic circumstellar environments.

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