

Numerical Characterization of Aperiodic Dynamics, Recurrence, and Initial-Condition Sensitivity in the Classical Lorenz System

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Abstract- This study presents a numerical investigation of nonlinear motion, recurrence, and predictability in the classical Lorenz system using a compact three-dimensional convection model. The analysis combines equilibrium considerations, local stability concepts, temporal integration, phase-space visualization, recurrence mapping, and sensitivity to nearby initial conditions. Numerical simulations show that the state variables remain bounded while developing irregular oscillations after an initial transient, with repeated reversals reflecting transitions between distinct regions of state space. The reconstructed three-dimensional trajectory exhibits the characteristic two-lobed attractor, while its planar projections reveal repeated circulation around the two non-zero equilibrium regions without convergence to a stationary state or stable periodic orbit. Successive local maxima form an organized nonlinear return relation, demonstrating that the apparently irregular oscillations retain an underlying deterministic structure. Sensitivity analysis further shows that initially close trajectories separate substantially with time, limiting reliable long-term prediction despite deterministic governing equations. The combined temporal, geometric, recurrence, and sensitivity analyses therefore provide a consistent computational characterization of bounded aperiodic motion in the Lorenz system. The results reinforce the importance of low-dimensional nonlinear models for understanding deterministic unpredictability and recurrent chaotic dynamics in mathematical physics.

Keywords— Lorenz system; nonlinear dynamics; strange attractor; recurrence; sensitivity to initial conditions; predictability; numerical simulation.

I. INTRODUCTION

Nonlinear dynamical systems provide a fundamental mathematical framework for describing physical processes in which deterministic governing equations may generate complex and apparently irregular motion (Lusch et al., 2018). The Lorenz system remains one of the most influential low-dimensional models because it demonstrates how a small set of nonlinear ordinary differential equations can produce bounded but nonperiodic dynamics (Li et al., 2020; Okomo et al., 2026). The uploaded source derives this three-mode convection model from a truncated representation of thermal

convection and subsequently examines its numerical evolution in phase space.

Recent investigations have shown that the classical Lorenz system possesses a richer attractor structure than is suggested by its conventional two-lobed representation (Monoran et al., 2025; Rastegin, 2024). Analytical techniques based on invariant sets and Lyapunov functions have also improved estimates of bounded regions containing Lorenz trajectories (Pinsky, 2023). Computational learning approaches have demonstrated that the geometry and dynamics of chaotic attractors can be reconstructed from observed trajectories. These developments indicate that attractor geometry,

boundedness, and trajectory recurrence remain important subjects in contemporary nonlinear dynamics (Margazoglou & Magri, 2023).

Predictability is another central issue arising from Lorenz-type systems. Recurrence analysis of generalized Lorenz models has demonstrated the coexistence of chaotic and non-chaotic solutions under different initial conditions (Shen, 2023). Modern atmospheric studies have consequently reconsidered the assumption that all solutions associated with nonlinear weather models exhibit identical chaotic characteristics (Shen, 2020; Shen et al., 2022). Numerical investigations of practical stability have further shown that parameter-dependent boundaries can separate qualitatively different dynamical regimes (Hochman et al., 2021). Lyapunov-based approaches remain particularly useful for characterizing stability and estimating the behavior of nonlinear Lorenz-type trajectories (Śliwiak & Wang, 2022).

Contemporary research has also focused on computational representations of chaotic dynamics. Reservoir-computing methods can reproduce Lorenz-type attractors and emulate important dynamical properties of the original nonlinear system (Röhm et al., 2021; Sudakow et al., 2022). Numerical approximation techniques continue to provide efficient means of investigating high-dimensional extensions of Lorenz dynamics (Kabayama et al., 2025). Data-driven prediction methods have also been examined as alternatives to purely equation-based forecasting of nonlinear trajectories. These studies show that computational analysis can complement classical dynamical-systems theory without modifying the governing mathematical model (Iliopoulos et al., 2024).

Generalized Lorenz formulations have broadened the interpretation of chaos, regularity, and predictability in nonlinear physical systems (Xavier et al., 2024). The coexistence of ordered and chaotic solutions in related models has motivated renewed investigation of the dependence of long-term dynamics on initial states (Shen et al., 2022). Coupled Lorenz systems have further been examined for synchronization and nonlinear information-transfer

phenomena (Khatun et al., 2024). Other studies have investigated the integrability of Lorenz-like systems to identify parameter regimes associated with regular and irregular motion (Seshadri & Lakshmivarahan, 2023). Fractional-order extensions have introduced memory effects while preserving the basic nonlinear interaction structure characteristic of Lorenz-type equations (Alhazmi & Saber, 2025; Tao et al., 2024).

Parameter-dependent dynamics continue to attract significant attention. Variations in system parameters can generate transitions among equilibrium, periodic, transient-chaotic, and sustained-chaotic responses (Sofroniou et al., 2025). Hybrid-order Lorenz formulations have similarly revealed complex attractor transitions and transport behavior (Yang & Liao, 2024). Stochastic versions of Lorenz-type systems demonstrate that nonlinear instability can interact strongly with intrinsic fluctuations (Khairallah et al., 2025; Леонов et al., 2015). New three-dimensional Lorenz-like systems are therefore commonly evaluated through equilibrium analysis, phase portraits, eigenvalues, and Lyapunov characteristics (Araz & Riaz, 2025; Qiao & Li, 2013).

The classical model itself also continues to generate new mathematical results. Rigorous analysis has established a rich organization of periodic trajectories embedded within Lorenz dynamics (Chimal-Eguía et al., 2025; Кузнецов et al., 2020). Thermodynamic interpretations have provided an alternative physical perspective on energy conversion within the Lorenz model (Katagiri et al., 2025; Ovsyannikov et al., 2023). Multiscale analysis of chaotic time series has further shown that nonlinear temporal information may be characterized beyond conventional phase-space plots (Li & Izumida, 2023). Short periodic orbits have recently been identified using geometric similarity techniques applied directly to Lorenz trajectories (Ghil & Lucarini, 2020). Bifurcation theory remains important for understanding transitions between qualitatively distinct Lorenz-type solutions. Recent analysis of Lorenz-like systems has identified double-zero bifurcations and associated changes in local dynamics (Creaser et al., 2015). Computational

trajectory classification has also been used to distinguish periodic, intermittent, transient, and chaotic responses (Araz & Riaz, 2025). Machine-learning methods have more recently been investigated for controlling transient chaotic behavior in the Lorenz system (Brunton et al., 2017; Valle et al., 2025). These developments demonstrate that the Lorenz equations remain relevant to mathematical physics, nonlinear science, computational mathematics, and predictability research.

The present study investigates the nonlinear behavior of the Lorenz system through a compact computational analysis of its temporal and phase-space dynamics. Numerical simulations are used to examine state evolution, attractor geometry, planar phase portraits, recurrence between successive extrema, and sensitivity to nearby initial conditions. Emphasis is placed on identifying the principal dynamical features associated with bounded aperiodic motion and limited predictability. The analysis provides a concise mathematical and computational representation of the system suitable for contemporary studies in nonlinear dynamics and mathematical physics.

II. METHODOLOGY

1. Mathematical Model

The study considers the reduced three-dimensional convection model obtained by Lorenz from a truncated Fourier representation of the two-dimensional thermal-convection equations. In the original paper, the amplitudes of the three retained modes are denoted by X , Y , and Z . The resulting nonlinear system is

$$\frac{dx}{dt} = \sigma(y - x), \quad (1)$$

$$\frac{dy}{dt} = x(\rho - z) - y, \quad (2)$$

$$\frac{dz}{dt} = xy - \beta z. \quad (3)$$

For the revised notation, $x(t)$, $y(t)$, and $z(t)$ replace the original capital variables X , Y , and Z . The parameter σ is retained because it remains the conventional notation for the Prandtl-number

parameter. The original parameter ris is replaced by ρ , while β is replaced by β , consistent with the notation commonly used for the Lorenz system today.

The three state variables retain the physical interpretation given by Lorenz: x represents the intensity of convective motion, y is associated with the horizontal temperature contrast between ascending and descending fluid, and z represents the distortion of the vertical temperature profile from the conductive state.

2. Relation to the Convection Model

Lorenz obtained the reduced system from the two-dimensional convection equations introduced by Saltzman. The original formulation uses the stream function ψ and the temperature departure θ , together with gravitational acceleration, thermal expansion, viscosity, thermal conductivity, layer depth, and imposed temperature difference.

In the updated presentation, the principal physical quantities may be denoted as

$$\Psi(x, z, t)$$

For the stream function and

$$\Theta(x, z, t)$$

for the departure of temperature from the conductive temperature profile.

Following the Fourier-mode truncation used in the source paper, only three time-dependent mode amplitudes are retained. These amplitudes are represented here by

$$x(t), y(t), z(t).$$

The dimensionless parameters are written as

$$\sigma = \frac{\nu}{\kappa}, \quad (4)$$

$$\rho = \frac{Ra}{Ra_c}, \quad (5)$$

and

$$\beta = \frac{4}{1 + k^2}, \quad (6)$$

where ν is the kinematic viscosity, κ is the thermal diffusivity, Ra is the Rayleigh number, Ra_c is its critical value, and k denotes the dimensionless horizontal wave-number parameter. In the original article these quantities are represented by σ , r , b , and a , respectively.

3. State-Space Representation

For numerical analysis, the system may be expressed compactly using the state vector

$$\mathbf{x}(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}. \quad (7)$$

The governing equations can then be represented as

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}; \boldsymbol{\mu}), \quad (8)$$

where

$$\boldsymbol{\mu} = (\sigma, \rho, \beta)$$

is the parameter vector and

$$\mathbf{f}(\mathbf{x}; \boldsymbol{\mu}) = \begin{bmatrix} \sigma(y - x) \\ x(\rho - z) - y \\ xy - \beta z \end{bmatrix}. \quad (9)$$

This vector form is preferable for the reconstructed paper because it simplifies numerical implementation in MATLAB or Python.

4. Equilibrium States

The equilibrium states are obtained by setting

$$\frac{dx}{dt} = \frac{dy}{dt} = \frac{dz}{dt} = 0.$$

The trivial equilibrium is

$$E_0 = (0, 0, 0). \quad (10)$$

For

$$\rho > 1,$$

two additional equilibria exist:

$$E_{\pm} = (\pm\sqrt{\beta(\rho-1)}, \pm\sqrt{\beta(\rho-1)}, \rho-1). \quad (11)$$

These correspond to the two steady-convection states identified in the original paper.

5. Linear Stability Analysis

The local stability of an equilibrium is examined using the Jacobian matrix

$$J(x, y, z) = \begin{bmatrix} -\sigma & \sigma & 0 \\ \rho - z & -1 & -x \\ y & x & -\beta \end{bmatrix}. \quad (12)$$

The eigenvalues of the Jacobian evaluated at each equilibrium determine whether perturbations decay or grow. Lorenz used this linearized formulation to determine the onset of convection and the loss of stability of the non-zero steady states.

For the classical parameter set used in the numerical part of the paper,

$$\sigma = 10, \beta = \frac{8}{3}, \rho = 28. \quad (13)$$

Lorenz notes that the critical value for instability of steady convection is approximately

$$\rho_c = 24.74,$$

and therefore selected $\rho = 28$ as a slightly supercritical value.

6. Numerical Results and Dynamical Analysis

The numerical solution of the Lorenz system exhibits an initial transient followed by irregular oscillatory behavior. The state variables $x(t)$, $y(t)$, and $z(t)$ remain bounded, but their oscillations do not settle into a simple repeating pattern. This behavior is consistent with the nonperiodic solutions reported in the original study.

$$\Delta t = 0.01. \quad (14)$$

The initial condition was chosen as a small perturbation from the no-convection equilibrium,

$$\mathbf{x}(0) = \begin{bmatrix} 0 \\ 0.1 \\ 0 \end{bmatrix}. \quad (15)$$

The source then followed the evolution of the three state variables for thousands of iterations and examined both their temporal behavior and their projections in phase space.

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Table 2. Lor Table 1: numerical solution for the first 160 iterations.

N	X (source print)	Y (source print)	Z (source print)
0000	0000	0010	0000
0005	0004	0012	0000
0010	0009	0020	0000
0015	0016	0036	0002
0020	0030	0066	0007
0025	0054	0115	0024
0030	0093	0192	0074
0035	0150	0268	0201
0040	0195	0234	0397
0045	0174	0055	0483
0050	0097	-0067	0415
0055	0025	-0093	0340
0060	-0020	-0089	0298
0065	-0046	-0084	0275
0070	-0061	-0083	0262
0075	-0070	-0086	0256
0080	-0077	-0091	0255
0085	-0084	-0095	0258
0090	-0089	-0098	0266
0095	-0093	-0098	0275
0100	-0094	-0093	0283
0105	-0092	-0086	0297
0110	-0088	-0079	0286
0115	-0083	-0073	0281
0120	-0078	-0070	0273
0125	-0075	-0071	0264
0130	-0074	-0075	0257
0135	-0076	-0080	0252
0140	-0079	-0087	0251
0145	-0083	-0093	0254
0150	-0088	-0098	0262
0155	-0092	-0099	0271
0160	-0094	-0096	0281

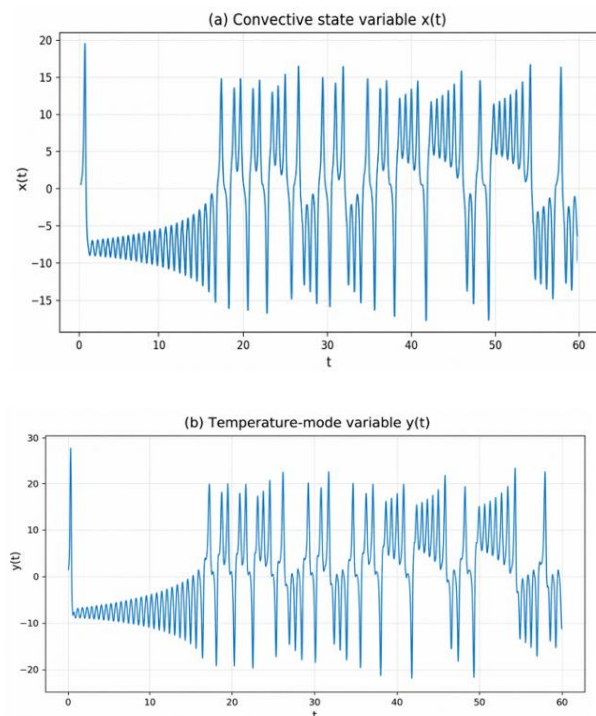
Source-table scaling. Lorenz states that, in the printed output (not in the computation), X, Y, and Z were multiplied by 10 and only the four figures to the left of the decimal point were printed. The

integer strings above are therefore reproduced exactly in the historical print convention. For example, 0084 corresponds to approximately 8.4 and 0270 to approximately 27.0.

1. Temporal Evolution

Starting from the small perturbation of the non-convective state, the solution initially exhibits rapid growth in the state variables. Lorenz observed an initial transient followed by oscillatory motion, with reversals in the sign of the convective variable as the trajectory moves between the two non-zero equilibrium regions. After an extended transient, the oscillations cease to follow a regular repeating sequence and the changes of sign occur at apparently irregular intervals.

For the updated analysis, the temporal histories of $x(t)$, $y(t)$, and $z(t)$ should be presented together rather than plotting only one variable. This provides a clearer view of the transient response, amplitude variation, and onset of irregular motion.



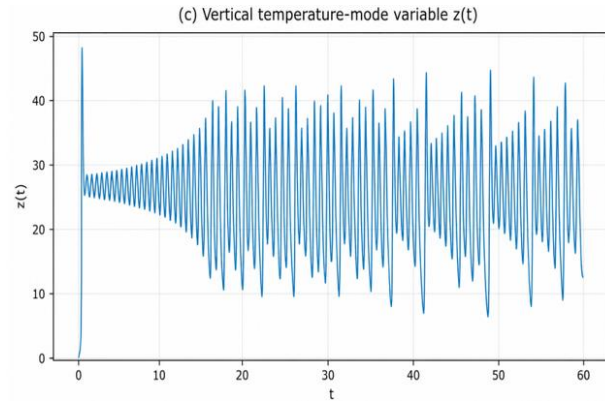


Figure 1. Temporal evolution of the Lorenz state variables.

Figure 1 shows that the amplitudes of the three state variables vary irregularly with time. In particular, the repeated changes in $x(t)$ and $y(t)$ indicate transitions between different regions of the system state space, while $z(t)$ remains oscillatory and positive over most of the computed interval.

Phase-Space Dynamics

The phase-space trajectory provides a clearer description of the nonlinear behavior. In the original analysis, Lorenz projected the trajectory onto different coordinate planes and showed that the motion repeatedly approaches the neighborhoods of the two non-zero steady states before moving from one region to the other. The trajectory does not settle onto either equilibrium and does not form a stable periodic orbit.

Using the revised notation, the two non-zero equilibria are

$$E_{\pm} = (\pm\sqrt{\beta(\rho-1)}, \pm\sqrt{\beta(\rho-1)}, \rho-1). \quad (17)$$

For the classical parameter values, the trajectory circulates around the neighborhoods of E_+ and E_- , producing the characteristic two-lobed phase-space structure associated with the Lorenz system.

A modern numerical presentation should combine the three-dimensional trajectory with its principal two-dimensional projections.

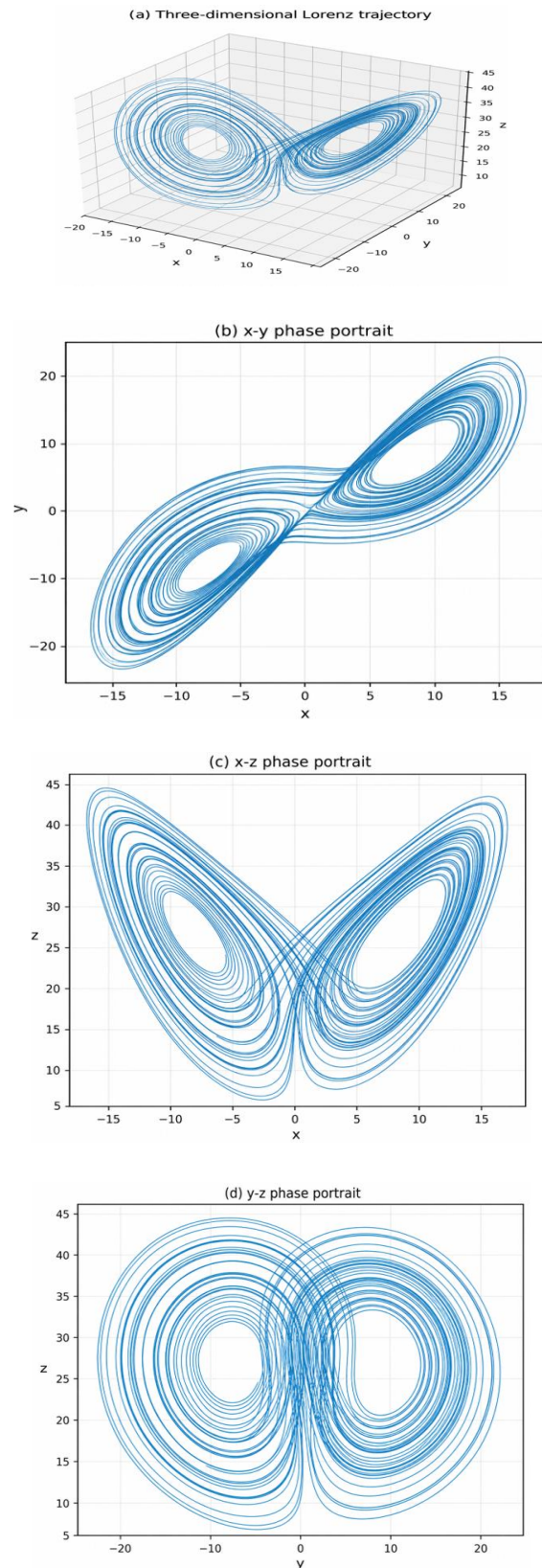


Figure 2. Phase-space structure and projections of the Lorenz attractor.

Figure 2 displays the characteristic two-lobed structure of the Lorenz trajectory. The three-dimensional attractor and its two-dimensional projections show repeated circulation around the two non-zero equilibrium regions. The irregular switching between these regions is a principal feature of the nonlinear dynamics of the system.

Aperiodicity and Return-Map Behavior

An important feature of the computed solution is that the trajectory remains bounded while failing to converge to either a stationary state or a simple periodic orbit. Lorenz examined successive extrema of the solution and developed a one-dimensional representation relating consecutive maxima. This analysis provided an early description of the complicated recurrent behavior of the system.

For the revised numerical analysis, let

$$z_1^{\max}, z_2^{\max}, \dots, z_n^{\max}$$

denote successive local maxima of $z(t)$. A return map can then be constructed as

$$z_{n+1}^{\max} = \mathcal{P}(z_n^{\max}), \quad (18)$$

where $\mathcal{P}$ denotes the numerically determined return relation.

Rather than retaining multiple separate diagrams of successive maxima, the updated paper can combine the return-map behavior and sensitivity analysis into one composite figure.

III. RECURRENCE AND PREDICTABILITY

To examine the recurrence behavior, let $z_n^{\max}$ denote the n -th local maximum of $z(t)$. Successive maxima may be represented through the return relation

$$z_{n+1}^{\max} = \mathcal{P}(z_n^{\max}), \quad (19)$$

where $\mathcal{P}$ denotes the numerical return map. Lorenz used successive maxima to demonstrate that the apparently irregular oscillations possess an underlying deterministic structure.

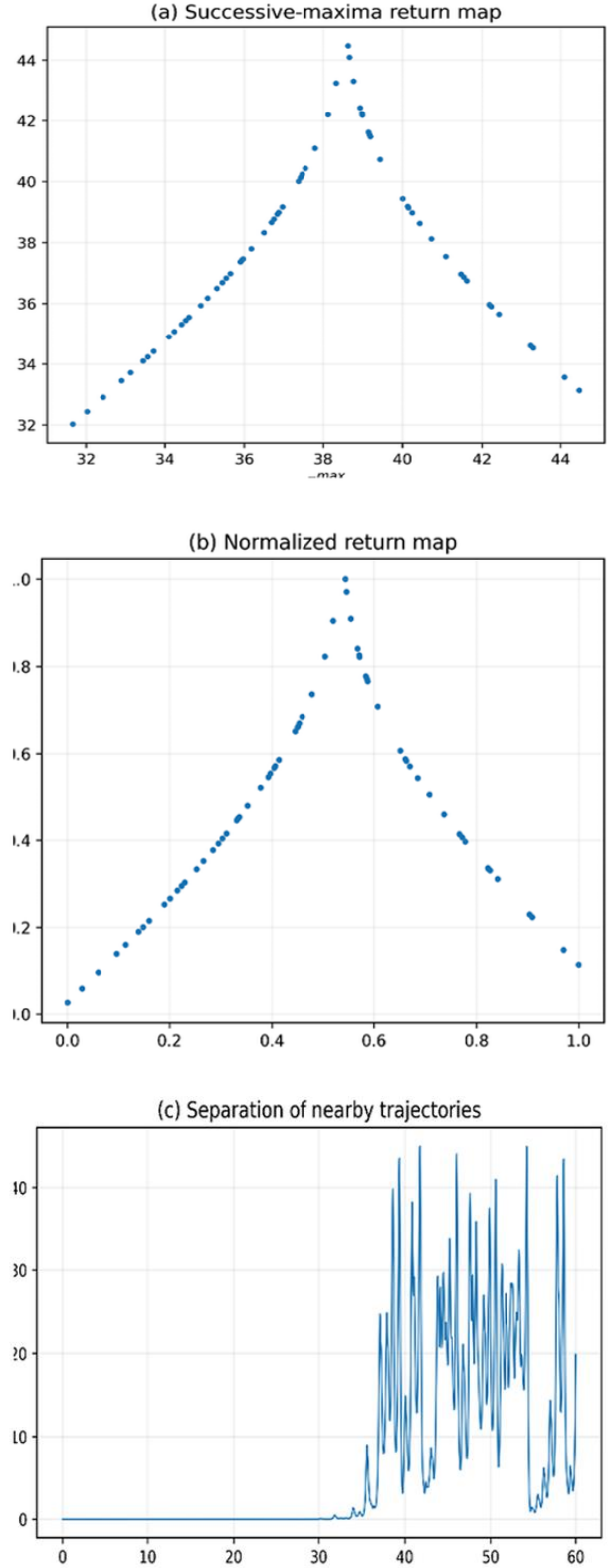


Figure 3. Return-map behavior and sensitivity of the Lorenz system to initial conditions.

The return map shows an organized nonlinear relationship between consecutive maxima rather than a random distribution of points. The additional nearby-trajectory comparison illustrates that small differences in the initial state can increase substantially with time. Thus, the system is deterministic but highly sensitive to its initial conditions, which restricts reliable long-term prediction.

Overall, the numerical results demonstrate that a low-dimensional deterministic system can generate bounded yet nonperiodic motion. The combination of irregular temporal oscillations, the two-lobed phase-space structure, and the nonlinear recurrence relation provides a consistent representation of the complex dynamics originally identified by Lorenz.

IV. DISCUSSION

The numerical results provide a coherent description of how deterministic nonlinear interactions generate bounded but irregular dynamics in the Lorenz system. The computed trajectories initially undergo a transient response before evolving into persistent oscillatory motion that does not approach a simple repeating sequence. The variables remain finite throughout the simulated interval, indicating that the irregularity is not associated with numerical divergence or unbounded growth. Instead, the behavior is produced by the internal nonlinear coupling of the three state variables. The repeated sign reversals of the convective component and corresponding changes in the temperature-related variables indicate continual redistribution of the trajectory between different regions of state space.

The phase-space analysis strengthens this interpretation. The three-dimensional trajectory forms the familiar two-lobed structure associated with Lorenz dynamics, and the planar projections show repeated circulation around the neighborhoods of the two non-zero equilibrium states. The orbit does not converge to either equilibrium and does not form a stable periodic cycle. Instead, it repeatedly leaves one lobe and enters the other in an apparently irregular sequence. This combination of confinement and continual switching is important because it distinguishes

chaotic bounded motion from both simple periodic oscillation and unstable divergence. The geometry therefore reflects simultaneous organization and instability: the trajectory remains restricted to a structured region of state space while individual transitions between the two lobes cannot be predicted indefinitely from finite-precision initial information.

The recurrence analysis provides additional evidence that the irregular oscillations are deterministic rather than random. Successive local maxima are not scattered arbitrarily but follow a recognizable nonlinear return relation. Such structure indicates that information about a current extremum constrains the range of the subsequent extremum, even though the resulting sequence does not settle into a simple periodic pattern. The return map therefore offers a reduced representation of the dynamics and complements the three-dimensional attractor by demonstrating that recurrent behavior is organized by an underlying deterministic mapping.

Sensitivity to nearby initial conditions is central to the predictability implications of the results. The numerical comparison of initially close trajectories shows that their separation remains small initially but later grows substantially. This divergence does not contradict determinism; rather, it demonstrates that deterministic systems can possess limited practical predictability. Small uncertainties in the initial state can eventually produce substantially different positions on the attractor. Consequently, accurate short-term evolution may coexist with poor long-term trajectory prediction.

Taken together, the temporal histories, phase-space geometry, recurrence structure, and nearby-trajectory divergence describe different aspects of the same nonlinear mechanism. Temporal plots reveal irregular oscillation, phase portraits demonstrate bounded geometrical organization, the return map exposes deterministic recurrence, and sensitivity analysis explains the loss of long-term predictive accuracy. The agreement among these independent numerical diagnostics strengthens the interpretation of the computed motion as bounded

aperiodic dynamics rather than numerical noise or an unresolved periodic solution.

From a computational perspective, the study also demonstrates the continuing value of the Lorenz system as a benchmark for nonlinear numerical analysis. Its low dimensionality permits direct visualization and relatively inexpensive simulation while retaining essential characteristics of more complicated nonlinear systems, including instability, recurrent motion, state-space organization, and restricted predictability. However, the present analysis is primarily geometrical and trajectory based. A more complete quantitative characterization would require complementary measures such as Lyapunov exponents, convergence tests, parameter continuation, and explicit numerical-error assessment. These additions would strengthen the distinction between qualitative evidence of sensitive dependence and quantitatively established chaotic instability.

V. CONCLUSION

A numerical analysis of the Lorenz system was conducted to examine temporal evolution, phase-space organization, recurrence, and sensitivity to nearby initial conditions. The simulations show that the system evolves from an initial transient into bounded irregular oscillations in which the state variables do not converge to a stationary state or simple periodic motion. The corresponding phase-space trajectory develops a structured two-lobed attractor characterized by repeated circulation around the two non-zero equilibrium regions and irregular transitions between them.

The return-map analysis shows that successive maxima possess an organized nonlinear relationship, demonstrating that the observed irregularity retains deterministic structure. The divergence of nearby trajectories further establishes the practical limitation of long-term prediction, since small initial differences can evolve into substantial trajectory separation. These findings collectively demonstrate that low-dimensional deterministic equations can generate motion that is simultaneously bounded,

structured, recurrent, and highly difficult to predict over long time intervals.

The study therefore provides a compact computational representation of the principal dynamical characteristics of the Lorenz system. Its main significance lies in showing how temporal behavior, attractor geometry, recurrence, and sensitivity complement one another in the interpretation of nonlinear aperiodic motion. The framework is suitable as a basis for further quantitative investigations of stability, chaotic intensity, parameter dependence, and predictability in nonlinear dynamical systems.

Recommendation

- Future work should compute the complete Lyapunov spectrum and related quantitative indicators so that sensitive dependence and phase-space contraction can be assessed numerically rather than only through trajectory comparison.
- The analysis should be extended through systematic variation of the governing parameters to identify equilibrium, periodic, transient, and chaotic regions and to establish how the attractor changes across critical parameter boundaries.
- Numerical robustness should be examined using different integration tolerances, step sizes, simulation lengths, and nearby initial states to confirm that the observed recurrence and sensitivity characteristics are independent of the chosen computational settings.

REFERENCES

1. Alhazmi, M., & Saber, S. (2025). APPLICATION OF FRACTAL-FRACTIONAL OPERATORS TO FRACTIONAL CHAOTIC SYSTEMS: INSIGHTS INTO COMPLEX DYNAMICS. *Fractals*, 34(2). <https://doi.org/10.1142/s0218348x25402534>
2. Araz, S. İ., & Riaz, M. B. (2025). Qualitative Behavior and Bifurcation Structure of a New Lorenz-type Chaotic Model. In *SPIRE - Sciences Po Institutional REpository*. <https://hal.science/hal-05175933>

3. Brunton, S. L., Brunton, B. W., Proctor, J. L., Kaiser, E., & Kutz, J. N. (2017). Chaos as an intermittently forced linear system. *Nature Communications*, 8(1), 19. <https://doi.org/10.1038/s41467-017-00030-8>
4. Chimal-Eguía, J. C., Guzmán-Aguilar, F., Silva-García, V. M., Báez-Medina, H., & Cardona-López, M. A. (2025). From Different Systems to a Single Common Model: A Review of Dynamical Systems Leading to Lorenz Equations. *Axioms*, 14(6), 465. <https://doi.org/10.3390/axioms14060465>
5. Creaser, J., Krauskopf, B., & Osinga, H. M. (2015). α -flips and T-points in the Lorenz system. *Nonlinearity*, 28(3). <https://doi.org/10.1088/0951-7715/28/3/r39>
6. Ghil, M., & Lucarini, V. (2020). The physics of climate variability and climate change. *Reviews of Modern Physics*, 92(3). <https://doi.org/10.1103/revmodphys.92.035002>
7. Hochman, A., Messori, G., Quinting, J., Pinto, J. G., & Grams, C. M. (2021). Do Atlantic-European Weather Regimes Physically Exist? *Geophysical Research Letters*, 48(20). <https://doi.org/10.1029/2021gl095574>
8. Kabayama, T., Komuro, M., Aihara, K., & Nakajima, K. (2025). Pseudo-Lorenz attractors in echo state networks. *Japan Journal of Industrial and Applied Mathematics*, 42(4), 1459–1473. <https://doi.org/10.1007/s13160-025-00740-3>
9. Katagiri, S., Matsuoka, Y., & Sugamoto, A. (2025). Applications of Nambu Non-equilibrium Thermodynamics to Specific Phenomena. In *ArXiv.org*. <https://doi.org/10.48550/arxiv.2509.12641>
10. Khairallah, G. M., Al-Hajj, S., Mowafi, H., Rahme, D., & Sakr, C. J. (2025). Occupational injury severity among healthcare workers: a retrospective study. *BMC Public Health*, 25(1), 1447. <https://doi.org/10.1186/s12889-025-22727-y>
11. Khatun, A. A., Muthanna, Y. A., Punetha, N., & Jafri, H. H. (2024). Collective dynamics of coupled Lorenz oscillators near the Hopf boundary: Intermittency and chimera states. *Physical Review E*, 109(3), 34208. <https://doi.org/10.1103/physreve.109.034208>
12. Li, H., Fernex, D., Semaan, R., Tan, J., Morzyński, M., & Noack, B. R. (2020). Cluster-based network model. *Journal of Fluid Mechanics*, 906. <https://doi.org/10.1017/jfm.2020.785>
13. Li, Z., & Izumida, Y. (2023). Thermodynamic efficiency of atmospheric motion governed by the Lorenz system. *Physical Review E*, 108(4), 44201. <https://doi.org/10.1103/physreve.108.044201>
14. Iliopoulos, A. P., Lunasin, E., Michopoulos, J. G., Rodriguez, S. N., & Wiggins, S. (2024). Data-Driven Model Identification Using Time Delayed Nonlinear Maps for Systems with Multiple Attractors. In *arXiv (Cornell University)*. Cornell University. <https://doi.org/10.48550/arxiv.2411.10910>
15. Lusch, B., Kutz, J. N., & Brunton, S. L. (2018). Deep learning for universal linear embeddings of nonlinear dynamics. *Nature Communications*, 9(1), 4950. <https://doi.org/10.1038/s41467-018-07210-0>
16. Margazoglou, G., & Magri, L. (2023). Stability analysis of chaotic systems from data. *Nonlinear Dynamics*, 111(9), 8799–8819. <https://doi.org/10.1007/s11071-023-08285-1>
17. Monoran, C. C., Watkins, C. E., & Breckling, S. (2025). An extension of the 3D Lorenz model under the Gay–Lussac approximation. *Chaos: An Interdisciplinary Journal of Nonlinear Science*, 35(9). <https://doi.org/10.1063/5.0288919>
18. Okomo, U., Fitzgerald, F., Feasey, N., Mehtar, S., & Dramowski, A. (2026). Infection prevention and control in low- and middle-income countries: policy, practice, and implementation challenges in sub-Saharan Africa. *Clinical Microbiology Reviews*, 39(2). <https://doi.org/10.1128/cmr.00142-22>
19. Ovsyannikov, I., Rademacher, J. D. M., Welter, R., & Lu, B. (2023). Time Averages and Periodic Attractors at High Rayleigh Number for Lorenz-like Models. *Journal of Nonlinear Science*, 33(5). <https://doi.org/10.1007/s00332-023-09933-x>
20. Pinsky, T. (2023). Analytical study of the Lorenz system: Existence of infinitely many periodic orbits and their topological characterization. *PubMed Central*, 120(31). <https://doi.org/10.1073/pnas.2205552120>

21. Qiao, Z., & Li, X. (2013). Dynamical analysis and numerical simulation of a new Lorenz-type chaotic system. *Mathematical and Computer Modelling of Dynamical Systems*, 20(3), 264–283.
<https://doi.org/10.1080/13873954.2013.824902>
22. Rastegin, A. E. (2024). A five-dimensional Lorenz-type model near the temperature of maximum density. *arXiv (Cornell University)*, 36(7).
<https://doi.org/10.1063/5.0216008>
23. Röhm, A., Gauthier, D. J., & Fischer, I. (2021). Model-free inference of unseen attractors: Reconstructing phase space features from a single noisy trajectory using reservoir computing. *Chaos: An Interdisciplinary Journal of Nonlinear Science*, 31(10), 103127.
<https://doi.org/10.1063/5.0065813>
24. Seshadri, A. K., & Lakshmivarahan, S. (2023). Invariants and chaos in the Volterra gyrostat without energy conservation. *Chaos, Solitons & Fractals*, 173, 113638.
<https://doi.org/10.1016/j.chaos.2023.113638>
25. Shen, B. (2020). Homoclinic Orbits and Solitary Waves within the Nondissipative Lorenz Model and KdV Equation. *International Journal of Bifurcation and Chaos*, 30(14), 2050257.
<https://doi.org/10.1142/s0218127420502570>
26. Shen, B. (2023). A Review of Lorenz's Models from 1960 to 2008. *International Journal of Bifurcation and Chaos*, 33(10).
<https://doi.org/10.1142/s0218127423300240>
27. Shen, B., Pielke, R. A., Zeng, X., Cui, J., Faghieh-Naini, S., Paxson, W., Kesarkar, A. P., Zeng, X., & Atlas, R. (2022). The Dual Nature of Chaos and Order in the Atmosphere. *Atmosphere*, 13(11), 1892. <https://doi.org/10.3390/atmos13111892>
28. Śliwiak, A. A., & Wang, Q. (2022). Approximating the linear response of physical chaos. *Nonlinear Dynamics*, 111(2), 1835–1869.
<https://doi.org/10.1007/s11071-022-07885-7>
29. Sofroniou, A., Martini, D. D., & Premnath, B. (2025). Perturbed Lorenz Attractors: Effects of Time-Dependent Disturbances on Chaotic Dynamics. *WSEAS TRANSACTIONS ON SYSTEMS AND CONTROL*, 20, 396–406.
<https://doi.org/10.37394/23203.2025.20.41>
30. Sudakow, I., Pokojovy, M., & Lyakhov, D. (2022). Statistical mechanics in climate emulation: Challenges and perspectives. *Environmental Data Science*, 1.
<https://doi.org/10.1017/eds.2022.15>
31. Tao, Y., Alhazmi, M., Youssif, M. Y., Elhag, A. E., Aljohani, A. F., & Saber, S. (2024). Analysis of a Lorenz model using adomian decomposition and fractal-fractional operators. *Thermal Science*, 28, 5001–5009.
<https://doi.org/10.2298/tsci2406001y>
32. Valle, D., Capeáns, R., Wagemakers, A., & Sanjuán, M. A. F. (2025). Controlling transient chaos in the Lorenz system with machine learning. *The European Physical Journal Special Topics*, 234(15), 3897–3905.
<https://doi.org/10.1140/epjs/s11734-025-01589-w>
33. Xavier, A. K., Demaeyer, J., & Vannitsem, S. (2024). Variability and predictability of a reduced-order land-atmosphere coupled model. *Earth System Dynamics*, 15(4), 893–912.
<https://doi.org/10.5194/esd-15-893-2024>
34. Yang, K., & Liao, X. (2024). Exploration of Parameter Space in the Lorenz Attractor. *Mathematical Modeling and Algorithm Application*, 3(2), 57–60.
<https://doi.org/10.54097/tw5qvr32>
35. Кузнецов, Н. В., Мокеев, Т. Н., Кузнецова, О. А., & Кудряшова, Е. В. (2020). The Lorenz system: hidden boundary of practical stability and the Lyapunov dimension. *Nonlinear Dynamics*, 102(2), 713–732.
<https://doi.org/10.1007/s11071-020-05856-4>
36. Леонов, Г. А., Кузнецов, Н. В., & Мокеев, Т. Н. (2015). Homoclinic orbits, and self-excited and hidden attractors in a Lorenz-like system describing convective fluid motion. *The European Physical Journal Special Topics*, 224(8), 1421–1458.
<https://doi.org/10.1140/epjst/e2015-02470-3>