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Study of Wake Evolution Behind Circular and Square Cylinders at Different Reynolds Numbers

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Abstract- Wakes of bluff bodies have long been considered a major area of interest within fluid dynamics due to their effect on drag production, pressure recovery, forces, vortex shedding, flow-induced vibrations, and wake mixing. Circular and square cylinders represent some canonical cases of bluff bodies that possess nearly similar features when it comes to the wake behavior; however, they significantly vary from each other concerning the separation process, growth of the near wake, and instabilities within the flow field. This paper offers an analysis of the evolution of wakes generated by both circular and square cylinders, focusing on the discussion of the mathematical and physical aspects related to the formation, development, and instability associated with the wake, vortex shedding, and the resulting force variation. Equations of incompressible flows are provided, while the evolution of the studied cases is analyzed via nondimensional numbers, including Reynolds number, Strouhal number, drag coefficient, and lift coefficient. Further mathematical analysis is performed via wall-stress-based separation criteria and perturbation-based concepts of instabilities. The findings show that for the circular cylinder, separation occurs through boundary-layer development and pressure gradients, while for the square cylinder, separation occurs through corners, which produce a wider wake and pressure deficits in the base region. With increasing Reynolds number, the flows on the two geometries evolve from steady, symmetric wakes to periodic, unsteady vortices, but the manner in which the flows do so is different owing to their shapes. It is concluded in the study that flow evolution in wakes depends upon inertia, viscosity, and geometric sharpness of the body shape, all of which need to be characterized mathematically.

Keywords: wake evolution, circular cylinder, square cylinder, bluff body flow, vortex shedding, Reynolds number, Strouhal number, hydrodynamic instability.

I. INTRODUCTION

Bluff body flow is an important subject in fluid mechanics due to both its applicability to engineering systems and its importance as a flow instability problem. For a flow encountering a blunt body, it becomes impossible for the flow to stay attached to the entire body surface. As a consequence, separation occurs, and this gives rise to a wake region with decreased momentum, and in most cases, significant unsteadiness. Issues involving mean drag, fluctuations in lift, sound production, heat transfer, mixing, and even structural responses to the above-mentioned issues affect bridge girders, towers, chimneys, risers, bundle tubes, urban flows, and electronic cooling systems. Of all bluff bodies considered, circular and square cylinders are the two most popular canonical configurations. The simplicity of their geometrical shapes provides enough scope to analyze their flow fields theoretically, numerically, or experimentally in a detailed manner. The flow characteristics observed during these studies also include several key elements found in any separated flows. While a circular cylinder with constant radius has a curved surface allowing the development of boundary layers followed by separation due to an adverse pressure gradient, a square cylinder with acute corner edges forces instantaneous separation of the flow.

The purpose of this paper is to examine the formation of wakes downstream of circular and square cylinders at various Reynolds numbers while incorporating mathematics in its discussion. This paper will cover topics such as governing equations, non-dimensionalization, separation criteria, wake scaling, instabilities, and force relationships. Through an integration of both physical interpretation and mathematical description, the goal is to gain a better understanding of how the geometry of bluff bodies affects wake formation.

II. GOVERNING EQUATIONS AND NONDIMENSIONAL FORMULATION

The fluid is assumed incompressible and Newtonian. The governing equations are the continuity equation and the Navier–Stokes equations:

$$\nabla \cdot u = 0$$
$$\rho\left(\frac{\partial u}{\partial t} + u \cdot \nabla u\right) = -\nabla p + \mu \nabla^2 u$$

where $u = (u, v, w)$is the velocity vector, p is pressure, ρ is density, and μ is dynamic viscosity.

For analysis of wake development using non-dimensionalization, the characteristic length will be the cylinder diameter or width (D), and the characteristic velocity will be the velocity of the free stream U_∞. The non-dimensionalized parameters are:

$$x^* = \frac{x}{D},$$
$$t^* = \frac{tU_\infty}{D},$$
$$u^* = \frac{u}{U_\infty},$$
$$p^* = \frac{p}{\rho U_\infty^2}$$

Substituting these into the governing equations yields

$$\nabla^* \cdot u^* = 0$$
$$\frac{\partial u^*}{\partial t^*} + u^* \cdot \nabla^{*u^*} = -\nabla^* p^* + \frac{1}{Re} \nabla^{*2u^*}$$

where the Reynolds number is

$$Re = \frac{\rho U_\infty D}{\mu}$$

This non-dimensional equation indicates that the main struggle will be between inertial transport and viscous diffusion. Low Re indicates the dominance of viscous stability while high Re indicates the dominance of inertia, resulting in separation, formation of shear layers, and instability.

The Strouhal number gives the frequency of vortex shedding f for unsteady wakes:

$$St = \frac{fD}{U_\infty}$$

The force coefficients are given by

$$C_D = \frac{F_D}{\frac{1}{2}\rho U_\infty^2 DL}$$
$$C_L = \frac{F_L}{\frac{1}{2}\rho U_\infty^2 DL}$$

where F_D and F_L are drag and lift forces, respectively, and L is the spanwise dimension.

This relation forms the mathematical basis for wake comparison of circular and square cylinders.

III. MATHEMATICAL DESCRIPTION OF SEPARATION AND WAKE FORMATION

The process of wake formation starts where the boundary layer/shear layer flow can no longer remain attached on the surface. In the case of a circular cylinder, the boundary layer separation results from the presence of an adverse pressure gradient. Assuming that s is the distance along the surface, boundary layer

$$\frac{dp}{ds} > 0$$

As the fluid near the wall loses momentum, the wall shear stress decreases. Separation occurs when the tangential wall shear stress vanishes:

$$\tau_w = \mu\left(\frac{\partial u_t}{\partial n}\right)_{wall} = 0$$

where u_t is the tangential velocity and n is the wall-normal direction. After this point, reversed flow begins to develop near the surface and the boundary layer separates.

In the case of the square cylinder, separation arises more from geometry than from viscosity. With its sharp corner at the leading edge, there is a sudden change in direction which the fluid flow cannot accommodate without separating. As such, separation starts right at the front corners of the body. Here, the separation point is more a result of the sudden geometry change than a gradual decrease in shear stress.

After separation takes place, a wake develops downstream of the body. The recirculation length $L_r$is one parameter that describes the wake length and is often normalized as

$$L_r^* = \frac{L_r}{D}$$

Recirculation length can be described as the downstream length from the rear stagnation zone to the point where the mean streamwise velocity goes back to zero. For both circular and square cylinders, L_r^* increases with Reynolds number at steady state; however, the rate of increase and the magnitude vary with cylinder shape.

The pressure coefficient along the surface of the body is

$$C_p = \frac{p - p_\infty}{\frac{1}{2}\rho U_\infty^2}$$

Pressure distribution decides the amount of form drag. Lower base pressure at the wake zone leads to higher pressure drag. Base suction is greater for square geometries; thus, they usually have higher values of drag coefficient compared to circular geometry.

IV. DEPENDENCE OF WAKE DEVELOPMENT ON REYNOLDS NUMBER

The wake development process is a function of Reynolds number. For very low Reynolds number values, the fluid flows steadily without any separation. Thus, the flow field around the body is symmetric and viscous forces overcome any inertial forces. Thus, if wake exists, it is weaker and stable.

As the Reynolds number increases, the flow field starts developing a steady recirculating wake. Such a wake comprises two vortices that occur behind the body symmetrically. With such wakes, the geometry plays an important role in deciding the wake width. The square cylinder creates a wider wake due to fixed separation while the circular cylinder creates a narrow wake.

This happens at a certain Reynolds number when the steady wake starts being unstable, leading to a periodic shedding of vortices. In mathematical terms, this can be viewed as a change from a steady state solution to an oscillatory solution of the Navier-Stokes equations. The wake ceases to have top-to-bottom symmetry and starts shedding vortices alternatively.

As the Reynolds number increases, the wake becomes more complicated until finally becoming three-dimensional. The spanwise variations, the streamwise vortices, and turbulence develop. While there is a similarity in the general process of transition for both the geometries, the square cylinder generally shows more shear-layer effects, as well as changes in wake width, due to more sudden separation.

V. LINEAR INSTABILITY CONCEPT

To understand the onset of vortex shedding mathematically, the flow may be decomposed into a steady base flow U and a disturbance u':

$$u(x,t) = U(x) + u'(x,t)$$

and similarly for pressure,

$$p(x,t) = P(x) + p'(x,t)$$

Substituting into the Navier–Stokes equations and neglecting quadratic disturbance terms yields a linearized system for the perturbations. Assume a modal disturbance of the form

$$u'(x,t) = \hat{u}(x)e^{\sigma t}$$

where $\sigma = \sigma_r + i\sigma_i$ is a complex eigenvalue. If

$$\sigma_r < 0$$

the disturbance decays and the steady wake is stable. If

$$\sigma_r > 0$$

the disturbance grows, and the wake becomes unstable. The oscillation frequency is related to the imaginary part:

$$f = \frac{\sigma_i}{2\pi}$$

The reason is that the occurrence of vortex shedding occurs when there is a critical Reynolds number above which the disturbances grow in strength and cause a self-sustaining, unsteady flow of the wake. Physically, the detached layers have inflectional velocity distributions, which can be unstable. The shape of the object is what dictates the nature of these layers. For circular objects, the layers occur due to boundary layer separation, while for square objects, the layers originate from the sharp corners on the surface.

VI. FORCE DEVELOPMENT AND OSCILLATORY LOADING

The forces acting on a cylinder arise from pressure and viscous stresses integrated over the body surface. The total drag and lift may be written as

$$F_D = \int_S (-pn_x + \tau_x)\, dS$$

$$F_L = \int_S (-pn_y + \tau_y)\, dS$$

where n_x and n_y are surface-normal components and τ_x, τ_y are shear contributions. For bluff bodies at moderate Reynolds number, pressure drag is usually dominant.

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In the steady symmetric regime, the mean lift is zero:

$$\overline{C_L} = 0$$

When vortex shedding begins, the lift becomes time-dependent and approximately oscillatory:

$$C_L(t) \approx A_L \sin(2\pi f t + \phi)$$

where A_L is the lift amplitude, while ϕ represents the phase constant. Also, the drag coefficient oscillates, usually around two times the shedding frequency:

$$C_D(t) \approx \overline{C_D} + A_D \cos(4\pi f t + \phi_D)$$

The reason for this is that there are two events that generate significant drag during each shedding cycle.

The average drag coefficient is higher in the case of the square cylinder since sharp separation creates lower base pressure. The circular cylinder tends to have lower average drag compared to similar Reynolds numbers due to separation downstream.

When the shedding frequency comes close to a natural frequency f_n,

$$f \approx f_n$$

Large-amplitude vibration is possible. This is a classical problem known as vortex-induced vibration, and it is very significant in the design of chimneys, towers, cables, and heat exchanger tubes.

VII. WAKE SCALES AND STRIPPING CHARACTERISTICS

A useful time scale in the motion of the wake is the shedding period

$$T = \frac{1}{f}$$

and a corresponding length scale is the convective distance traveled in one shedding period:

$$\lambda = \frac{U_\infty}{f}$$

This wake wavelength can also be written in terms of Strouhal number:

$$\lambda = \frac{D}{St}$$

Therefore, when the Strouhal number is known, one can calculate the spacing between the larger vortices formed in the wake.

The second scaling relationship is between the convection and diffusion processes in the wake. The timescale for convection is approximately

$$t_c \sim \frac{D}{U_\infty}$$

while the viscous diffusion timescale is

$$t_\nu \sim \frac{D^2}{\nu}$$

where $\nu = \mu/\rho$ is kinematic viscosity. Their ratio is

$$\frac{t_\nu}{t_c} \sim Re$$

which shows again why higher Reynolds number leads to wake structures that are less damped by viscosity and more prone to instability.

VIII. COMPARATIVE WAKE PHYSICS OF CIRCULAR AND SQUARE CYLINDERS

The distinction between circular and square cylinders is not merely in their geometry but in the nature of their flow-separation problem. In the case of the circular cylinder, the angle of separation depends on Reynolds number and surface roughness. Consequently, the dimensions of the wake region and its behavior are sensitive to changes in operating parameters. The near wake is strongly affected by the location of the zero-shear stress point, $\tau_w = 0$, and the boundary layer development prior to separation.

In the case of the square cylinder, the points of separation are essentially determined by the front corners. The wake will then depend less on the fine boundary layer development process and more on the subsequent behavior of the two free shear layers.

A very simple relation for mean drag can be regarded as

$$\overline{C_D} \propto C_{p,front} - C_{p,base}$$

Since $C_{p,base}$ is less positive in the case of the square cylinder, the drag will be larger. Likewise, wake width W_w can serve as an index for the base pressure deficiency and vortices strength. The square cylinder will always fulfill

$$W_{w,square} > W_{w,circular}$$

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Under similar conditions of low to moderate Reynolds numbers.

Both bodies display alternating vortex streets under shedding condition, although the square cylinder body might display pronounced shear layer flapping effect leading to sudden vortex shedding since its shear layers come into existence abruptly at the corners, while the circular cylinder is expected to create smoother vortices.

IX. THREE-DIMENSIONALITY AT HIGHER REYNOLDS NUMBER

At higher Reynolds numbers, the wake ceases to remain purely two-dimensional. A more general disturbance form must include the spanwise coordinate z:

$$u'(x,y,z,t) = \hat{u}(x,y)e^{i\beta z+\sigma t}$$

with β being the spanwise wavenumber. The growth of these three-dimensional instabilities results in spanwise waviness in the wake, streamwise vortices, and low wake coherence over the length of the cylinder.

It is crucial to note that force coherence over the span plays a significant role in determining the overall load on the structure. A completely coherent wake results in higher force fluctuations; however, incoherent wakes result in lower force fluctuation even if vortex shedding continues at a high rate locally.

Both the square and circular cylinders undergo three-dimensional instabilities eventually; however, their characteristics are highly dependent on the shape of the cylinder. Sharp corners in the square cylinder could enhance shear-layer instabilities, whereas smooth separation in the circular cylinder could postpone certain instabilities.

X. ENGINEERING IMPLICATIONS

Differences in mathematical and physical characteristics of circular and square cylinder wakes are of significant engineering value. The latter would normally create higher pressure drag that may not be desirable in certain aerodynamic and hydrodynamic applications. The square cylinder can also generate strong fluctuating forces on account of its sharper and wider shear layers. While circular cylinders might usually be superior regarding their mean drag, they may still be prone to vortex shedding and vibrations. This is particularly important in the design of towers, bridges, tube bundles, underwater bodies, and sensor housings. The shape itself, corner rounding, splitter plates, fairings, or other passive devices may be employed to alter the wake and, therefore, the separation process. It is possible to describe the basic dynamics in question mathematically through Reynolds and Strouhal numbers as well as force coefficients.

XI. CONCLUSION

In this paper, we have demonstrated a research-like paper with mathematics embedded in it for analyzing wake development behind circular and square cylinders with different Reynolds numbers. It was demonstrated that wake development occurs due to incompressible Navier-Stokes equation, which governs the flow, and the wake becomes steady or unstable based on Reynolds number, being the only dimensionless parameter involved in the problem. Strouhal number defines vortex shedding behavior, and drag coefficient and lift coefficient relate to predicting forces for engineering purposes. Mathematical considerations reveal several critical factors. First, separation behind the circular cylinder happens when wall shear stress becomes zero under the effect of adverse pressure gradient, but separation behind square cylinder is mainly caused by corners. Second, wake instability is explained using the concept of disturbances and their development. If the disturbance growth rate changes sign, the steady wake loses stability leading to periodic shedding. Lastly, forces oscillate due to vortex alternating, and Reynolds number affects scaling relations of wake unsteadiness. It can therefore be deduced from the above comparison between the two geometries that the body shape is important in determining the nature of the wake formation. For instance, circular cylinders typically experience delayed separation, thinner wakes, and reduced drag compared to square cylinders which, because of their separation at sharp edges, lead to the production of thicker wakes and suction force at the base and tend to have increased drag. Both geometries go through a similar series of development in the order of steady wake flow, periodic shedding, and eventually three-dimensional behavior as Reynolds number increases. In summary, the overall observation is that the wake flows behind bluff bodies cannot be explained based on the Reynolds number alone. The body geometry plays an important role in affecting the separation process, wake generation, instabilities, and forces.

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