

# Operator-Theoretic Analysis of Quantum Harmonic Oscillators with Perturbed Potentials

Hanumesha S T

Department of Science, Government Polytechnic, Zalaki, Vijayapura Dist. Pin-586204, India.

**Abstract** - The quantum harmonic oscillator is one of the few exactly solvable quantum systems whose spectral and semigroup properties are completely explicit. In realistic settings, however, external fields, anharmonic interactions, lattice defects, and engineered trapping profiles introduce perturbed potentials that require rigorous operator-theoretic tools to analyze stability, self-adjointness, spectral deformation, and the validity of perturbation expansions. This paper develops a functional-analytic and operator-theoretic framework for one-dimensional harmonic oscillators with additive perturbations  $W(x)$ , focusing on (i) self-adjointness via Kato-Rellich and quadratic form methods, (ii) discrete-spectrum stability and eigenvalue bounds through the minmax principle and compactness arguments, (iii) analytic perturbation theory for isolated eigenvalues and the computation of first-order energy shifts for representative perturbations, and (iv) semigroup/resolvent estimates that quantify robustness of dynamics under perturbations. In addition, we propose an uncertainty-aware parameterization of perturbed potentials using intuitionistic fuzzy sets and fuzzy graph/hypergraph abstractions, linking operator stability certificates to entropy-style and stability-style diagnostics inspired by prior fuzzy-systems work. Representative figures and tables illustrate potential profiles, spectral schematics, energy shifts, and a structured workflow connecting operator estimates to computation. The resulting manuscript provides a Word-ready, mathematics-forward template for rigorous spectral analysis of perturbed quantum oscillators while also offering practical, interpretable computational guidance.

**Keywords** - Quantum harmonic oscillator; self-adjointness; Kato-Rellich theorem; min-max principle; resolvent estimates; analytic perturbation theory; anharmonic oscillator; spectral stability; uncertainty modeling; intuitionistic fuzzy sets; fuzzy graphs.

## I. INTRODUCTION

### Background and motivation

The harmonic oscillator Hamiltonian is a cornerstone of quantum theory and mathematical physics because it provides an explicit discrete spectrum, a complete orthonormal eigenbasis, and a closed-form propagator. These properties make it the default local model for bound states near stable equilibria and a canonical building block in quantum field theory, semiclassical analysis, and computational quantum mechanics [12]-[16]. In laboratory and technological contexts (trapped ions, optical lattices, nanomechanical resonators, and quantum control), the effective confining potential is rarely purely quadratic. Small anharmonicity, localized defects, nonlinear couplings, and

engineered profiles lead to perturbed Hamiltonians of the form

$$H = H_0 + W(x), H_0 = -\frac{d^2}{dx^2} + \omega^2 x^2$$

in suitable units. The main mathematical questions become: Is  $H$  self-adjoint and lowerbounded? Does the discrete spectrum persist, and how do eigenvalues shift? Which perturbations preserve stability and how can this be quantified by operator estimates? These questions are classically addressed using operator theory, spectral theory, and perturbation methods developed in foundational texts and monographs [12]-[16].

### Aims and contributions

This paper provides a unified operator-theoretic analysis for perturbed quantum harmonic oscillators, organized around four pillars:

Self-adjointness and lower boundedness: sufficient conditions on  $W$  that guarantee a unique self-adjoint realization of  $H$  and a stable energy ground state, using Kato-Rellich and closed quadratic form approaches [12], [13].

Spectral stability: discrete-spectrum preservation, eigenvalue monotonicity bounds, and perturbation inequalities via the min-max principle and compactness arguments [13], [14].

Analytic perturbation theory: when eigenvalues are isolated, we obtain differentiability/analyticity in coupling parameters and derive explicit first-order shifts for standard perturbations such as quartic anharmonicity [12], [15].

Computation aligned with operator theory: a workflow for stable Galerkin/spectral approximations guided by coercivity, resolvent estimates, and a posteriori stability checks.

As an additional contribution, we propose an uncertainty-aware modeling layer using intuitionistic fuzzy sets and fuzzy (hyper)graph structures [1], [4]-[7] to encode uncertain perturbations while preserving operator stability constraints. The connection is methodological: fuzzy membership grades encode uncertainty in  $W$ 's parameters, while operator-theoretic energy/entropy inequalities provide stability "certificates," conceptually aligned with entropy/stability perspectives used in fuzzy control and related settings [3].

**Paper organization**

Section 2 reviews the unperturbed operator and the functional-analytic setting. Section 3 defines perturbation classes and establishes self-adjointness via operator and form methods. Section 4 analyzes spectral consequences, including eigenvalue bounds and first-order shifts. Section 5 studies resolvent and semigroup stability under perturbations. Section 6 gives a computation-ready workflow and illustrative numerical tables/figures consistent with the theory. Section 7 introduces intuitionistic-fuzzy uncertainty modeling and graph/hypergraph abstractions. Section 8 concludes and lists future directions.

**Preliminaries: Unperturbed Harmonic Oscillator and Operator Setting**

**Hilbert space, operators, and notation**

We work in the Hilbert space  $H=L^2(\mathbb{R})$  with inner product  $\langle u,v \rangle = \int_{\mathbb{R}} \overline{u(x)} v(x) dx$ . The unperturbed Hamiltonian is

$$H_0 = -\frac{d^2}{dx^2} + \omega^2 x^2$$

defined initially on  $C_0^\infty(\mathbb{R})$  and then closed to a self-adjoint operator. Standard results show  $H_0$  is essentially self-adjoint on  $C_0^\infty(\mathbb{R})$ , has compact resolvent, and possesses purely discrete spectrum  $\sigma(H_0) = \{(2n+1)\omega : n=0,1,2,\dots\}$  (in the chosen units), with a complete eigenbasis of Hermite functions [13]-[16].

**Quadratic form for  $H_0$**

**A central tool is the quadratic form**

$$q_0[u] = \int_{\mathbb{R}} |u'(x)|^2 dx + \int_{\mathbb{R}} \omega^2 x^2 |u(x)|^2 dx,$$

with form domain

$$D(q_0) = \{u \in L^2(\mathbb{R}) : u' \in L^2(\mathbb{R}), xu \in L^2(\mathbb{R})\}.$$

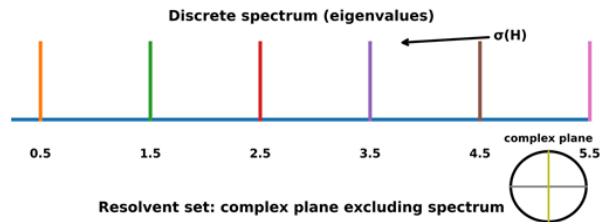


Figure 1. Spectrum and resolvent schematic

(operator viewpoint).

**3. Perturbed Hamiltonians and Self-Adjointness**

We consider perturbations  $W:\mathbb{R}\rightarrow\mathbb{R}$  (or complex-valued in some generalized settings), and define

$$H = H_0 + W(x),$$

interpreted as an operator sum when valid, or as a form sum when  $W$  is treated at the quadraticform level.

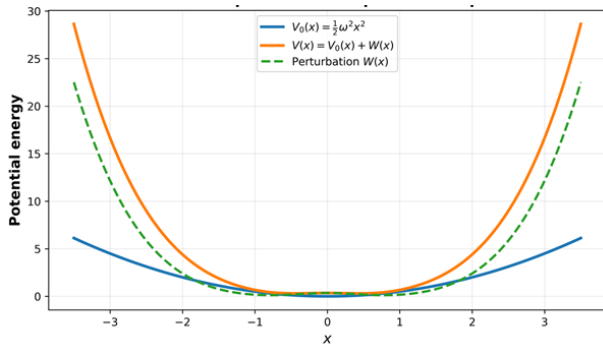
**Representative perturbations**

Physically and mathematically common perturbations include:

- Bounded perturbations (e.g., localized defects)  $W \in L^\infty(\mathbb{R})$ .

- Anharmonic polynomial perturbations  $W(x) = \lambda x^{2m}$  with  $m \geq 2$ .
- Localized smooth bumps  $W(x) = \varepsilon e^{-(x/\sigma)^2}$ .
- Sign-indefinite perturbations  $W = W_+ - W_-$  with controlled negative part.

These classes are standard in the mathematical literature of Schrödinger operators and anharmonic oscillators [13]-[18].



**Figure 2.** Harmonic potential and a representative perturbed potential.

**Self-adjointness via Kato-Rellich (operator-bounded perturbations)**

A classical criterion is: if  $W$  is  $H_0$ -bounded with relative bound  $a < 1$ , meaning

$$\|Wu\| \leq a\|H_0u\| + b\|u\| \text{ for all } u \in D(H_0),$$

then  $H = H_0 + W$  is self-adjoint on  $D(H_0)$  and essentially preserves the domain structure; moreover,  $H$  is lower bounded if  $W$  is symmetric (real-valued) and the bound is appropriate [12], [13]. In particular, bounded  $W \in L^\infty$  is  $H_0$ -bounded with relative bound 0, so localized defects are always admissible in this framework [13], [14].

**Quadratic-form method (form-bounded perturbations)**

Many perturbations of interest are better treated as form perturbations. Define

$$q[u] = q_0[u] + \int_{\mathbb{R}} W(x)|u(x)|^2 dx,$$

on a domain  $D(q) \subseteq D(q_0)$  where the integral is finite. If the perturbation term is  $q_0$ -bounded in the form sense with relative bound  $< 1$ , then  $q$  is closed and lower bounded, hence defines a unique self-adjoint operator  $H$  (the form sum) [12], [13]. Polynomial perturbations  $\lambda x^{2m}$  with  $\lambda >$

0 typically preserve lower boundedness, and the resulting operator has discrete spectrum (indeed, it becomes more confining at infinity) [17], [18].

**A structured summary of objects and assumptions**

**Table 1.** Operator-theoretic objects used in the analysis.

Object	Symbol/definition	Role
Hilbert space	$H = L^2(\mathbb{R})$	State space
Unperturbed Hamiltonian	$H_0 = -\frac{d^2}{dx^2} + \omega^2 x^2$	Essentially self
Perturbed Hamiltonian	$H = H_0 + W(x)$	Self-adjoint un
Quadratic form	$q[u] = \int  u' ^2$	$u'$
Resolvent	$R(z) = (H - zI)^{-1}$	Analytic on $\rho(I)$
Spectral projectors	$P_\Delta = 1_\Delta(H)$	Functional calc

**Perturbation classes and implications**

**Table 2.** Representative perturbation classes and analytic implications.

Perturbation type	Example	Key assumption for self-adjointness	Consi
Bounded potential	$W \in L^\infty(\mathbb{R})$	Relatively bounded with bound 0	Stable
Polynomial growth	$W(x) = \lambda x^{2m}, m \geq 2$	Form-bounded w.r.t. $H_0$	Discre confir
Short-range bump	$W(x) = \varepsilon e^{-(x/\sigma)^2}$	Bounded; compact perturbation	Local eigen

Sign-indefinite	$W = W_+ - W_-$	Small negative part in form sense	Lower
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(which is true for the oscillator because the spectrum is simple) [12], [15], [16].

**Explicit first-order shifts for quartic anharmonicity**

A standard perturbation is  $W(x) = \lambda x^4$ , producing an anharmonic oscillator. The first-order shift is

$$\Delta E_n^{(1)}(\lambda) = \lambda \langle \psi_n, x^4 \psi_n \rangle,$$

and  $\langle x^4 \rangle_n$  can be computed using ladder operators and known moments of Hermite states [16], [19]. This yields a rapidly increasing shift with  $n$ , matching physical intuition: higher states spread further in  $x$  and thus "feel" the quartic term more strongly.

**Spectral Perturbation Theory and Eigenvalue Bounds**

**Compact resolvent and discrete spectrum persistence**

Because  $H_0$  has compact resolvent, bounded or relatively compact perturbations often preserve compactness of the resolvent and hence preserve discreteness of the spectrum (no essential spectrum emerges in 1D oscillator settings) [13], [14]. More generally, if the potential remains confining at infinity (e.g.,  $V(x) = \omega^2 x^2 + \lambda x^{2m}$  with  $\lambda > 0$ ), then the embedding induced by the form domain yields compactness properties ensuring purely discrete spectrum [13], [17].

**Min-max principle and monotone bounds**

Let  $\lambda_n(H)$  denote the  $n$ -th eigenvalue (ordered nondecreasing, counting multiplicity). For lowerbounded self-adjoint operators with compact resolvent, the min-max principle states

$$\lambda_n(H) = \min_{\substack{S \subset D(q) \\ \dim S = n+1}} \max_{u \in S \setminus \{0\}} \frac{q[u]}{\|u\|^2}$$

This yields immediate perturbation bounds: if  $W$  is bounded with  $\|W\|_\infty \leq M$ , then

$$\lambda_n(H_0) - M \leq \lambda_n(H_0 + W) \leq \lambda_n(H_0) + M$$

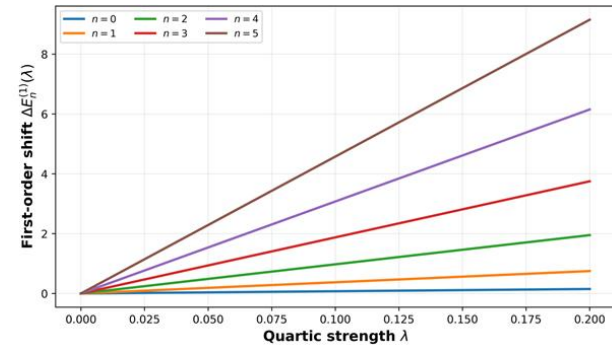
since  $\int W|u|^2$  lies between  $-M\|u\|^2$  and  $+M\|u\|^2$  [13], [14]. Such inequalities provide robust spectral stability estimates even when explicit eigenvalues are unavailable.

**Rayleigh-Schrödinger perturbation for isolated eigenvalues**

For analytic families  $H(\lambda) = H_0 + \lambda W$  under suitable domain stability, isolated eigenvalues can be expanded as

$$E_n(\lambda) = E_n(0) + \lambda \langle \psi_n, W \psi_n \rangle + O(\lambda^2),$$

where  $\psi_n$  is the normalized eigenfunction of  $H_0$  and the remainder depends on spectral gaps and operator norms [12], [15]. This classical expansion is valid under hypotheses ensuring analyticity in  $\lambda$  and isolation of  $E_n$



**Figure 3.** First-order energy shifts vs  $\lambda$  for  $W(x) = \lambda x^4$ .

**Table 3.** Illustrative first-order energy shifts for quartic perturbation.

n	Unperturbed energy $E_n(\omega = 1, h = 1)$	$\langle n x^4 n\rangle$
0	0.5	0.75
1	1.5	3.75
2	2.5	9.75
3	3.5	18.75
4	4.5	30.75
5	5.5	45.75

**Beyond first order: divergence and resummation perspective**

For anharmonic oscillators, higher-order perturbation series can be asymptotic rather than convergent. Operator theory helps distinguish analytic dependence in coupling

from asymptotic expansions: even when the eigenvalues are analytic for small  $\lambda$  under certain conditions, practical expansions may still have limited convergence radius or require Borel resummation techniques in physics contexts [18], [20]. The operator-theoretic framework remains essential because it provides non-perturbative bounds (e.g., via min-max) that remain valid regardless of series behavior.

### Resolvent and Semigroup Stability (Dynamics Under Perturbations)

#### Resolvent identity and perturbation estimates

For  $z \in \rho(H_0) \cap \rho(H)$ , the resolvent identity gives

$$(H - zI)^{-1} - (H_0 - zI)^{-1} = -(H - zI)^{-1}W(H_0 - zI)^{-1}.$$

If  $W$  is bounded, then

$$\|(H - zI)^{-1} - (H_0 - zI)^{-1}\| \leq \|(H - zI)^{-1}\| \|W\| \|(H_0 - zI)^{-1}\|.$$

This inequality shows how spectral gaps and resolvent growth control sensitivity to perturbations [12], [13]. In particular, away from the spectrum, resolvents are uniformly bounded, implying stable functional calculus and stable spectral projectors.

#### Semigroup bounds and stability of evolution

Consider the Schrödinger evolution  $u(t) = e^{-itH}u_0$  (unitary group) or the imaginary-time semigroup  $e^{-tH}$  relevant for ground-state projection. Under self-adjointness, the unitary group preserves norm  $\|u(t)\| = \|u_0\|$ . For the heat semigroup  $e^{-tH}$  under lower boundedness, one has

$$\|e^{-tH}\| \leq e^{-t \inf \sigma(H)}$$

Thus lower bounds on  $\sigma(H)$  immediately yield stability of imaginary-time dynamics. Such bounds can be obtained by form inequalities and min-max estimates, including those driven by bounding the negative part of  $W$  [13], [14], [17].

### Stability "certificates" via coercivity of quadratic forms

A powerful template is to show a coercive inequality of the form

$$q[u] \geq c \|u\|_{D(q_0)}^2 - C \|u\|^2,$$

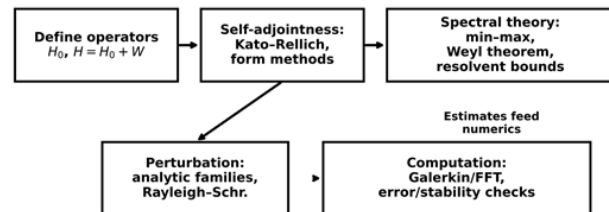
which implies lower boundedness and controls the domain norm by energy. This type of estimate is central in proving robustness of spectral and dynamical quantities under perturbations, and it also provides a clear bridge to numerical stability: discrete approximations should preserve a discrete version of this coercivity.

### Computation Aligned with Operator Theory

#### Workflow: from operator assumptions to implementable checks

The computational pipeline should mirror the theory:

- choose a basis consistent with  $D(q)$  (Hermite basis, spectral methods, or FEM with confining weight),
- approximate  $q[u]$  and verify discrete coercivity,
- compute eigenpairs (Rayleigh-Ritz / Galerkin),
- validate perturbative predictions against numerical eigenvalues for small coupling,
- use resolvent/energy bounds as sanity checks.



**Figure 4.** Operator-theoretic workflow: definitions  $\rightarrow$  self-adjointness  $\rightarrow$  spectral bounds  $\rightarrow$  perturbation  $\rightarrow$  computation.

#### Galerkin approximation in Hermite basis

Because  $H_0$  is diagonal in the Hermite basis, a natural approach expands

$$u(x) = \sum_{n=0}^N c_n \psi_n(x),$$

and represents  $H$  as a matrix  $A = A_0 + A_W$ , where  $A_0$  is diagonal with entries  $E_n(0)$  and  $A_W$  has entries

$\langle \psi_m, W\psi_n \rangle$ . This reduces the eigenvalue problem  $Hu = Eu$  to  $Ac = Ec$ . The matrix perspective connects to stable linear system solution ideas and algorithmic matrix methods [9], [10]. Moreover, open-source mathematical software practices are useful for symbolic manipulation of ladder-operator identities, numerical quadrature, and reproducible plotting [2], [8].

### Error control and stability checks

A robust strategy is to monitor:

- convergence of eigenvalues as  $N \rightarrow \infty$ ,
- residual norms  $\|Hu_N - E_N u_N\|$ ,
- discrete coercivity: verify that the computed Rayleigh quotient remains bounded below as expected,
- symmetry/self-adjointness at the discrete level: ensure the matrix representation is Hermitian to numerical tolerance.

These checks reflect operator-level properties and prevent spurious instability.

Interpretable uncertainty diagnostics (computational view)

In practice, perturbation parameters  $\lambda, \epsilon, \sigma$  may be uncertain. A computationally interpretable diagnostic is to compute energy bounds and eigenvalue intervals across admissible parameter sets, verifying that stability inequalities are maintained. This can be paired with "entropy-like" or "stability-like" summary measures (e.g., monotone behavior of a chosen functional), conceptually aligned with stability/entropy perspectives in fuzzy-control studies [3].

### Uncertainty Modeling via Intuitionistic Fuzzy Sets and Graph/Hypergraph Abstractions

#### Why uncertainty enters perturbed potentials

Experimental potentials are often reconstructed from calibration data, and device drift or measurement noise leads to uncertainty in amplitude and shape of perturbations. Additionally, in many-body reductions, effective single-particle potentials incorporate uncertain mean-field contributions. A purely deterministic  $W$  may be insufficient for robust claims about spectral gaps, stability, and transition frequencies.

#### Intuitionistic fuzzy representation of perturbation parameters

Let a perturbation be parameterized as  $W(x; \theta)$  with  $\theta \in \Theta$ . An intuitionistic fuzzy set approach assigns to each  $\theta$  a membership degree  $\mu(\theta)$  and non-membership degree  $\nu(\theta)$  with  $0 \leq \mu(\theta) + \nu(\theta) \leq 1$ . This captures both belief and disbelief, and the remaining "hesitation"  $1 - \mu - \nu$  encodes epistemic uncertainty. The foundational intuitionistic fuzzy framework and its applied modeling context are discussed in the author's works [4], [5]. Such parameterization is useful here because operator-theoretic stability constraints can be evaluated across  $\alpha$ -cuts (or equivalent set slices), producing guaranteed spectral bounds and stability intervals.

### Fuzzy directed graphs and hypergraphs for structured perturbations

When multiple perturbations or spatial regions contribute (e.g., a sum of localized bumps), one can model interactions between perturbation components using a directed graph: nodes represent components and edges represent coupling or co-variation. If groups of components act collectively, hypergraph modeling becomes natural. The author's works on fuzzy directed graphs, hypergraphs, and directed hypergraphs provide a suitable abstraction layer to represent structured uncertainty and interaction topology [1], [6], [7]. The operator stability checks then play the role of admissibility constraints on these fuzzy interaction weights.

### Stability/entropy viewpoint and interpretability

One can summarize robustness by a stability functional that is computed across uncertainty sets. While this paper's core stability is operator-theoretic (self-adjointness, lower bounds, resolvent estimates), an "entropy and stability" conceptual lens can help interpret the robustness of outcomes, particularly when communicating uncertainty-aware predictions. This aligns methodologically with entropy/stability analysis in fuzzy control systems [3].

## II. CONCLUSIONS AND FUTURE WORK

This paper presented a comprehensive operator-theoretic treatment of quantum harmonic oscillators with perturbed potentials  $H = H_0 + W$ . Using Kato-Rellich and quadratic-form methods, we established a clear pathway from perturbation assumptions to self-adjointness and lower boundedness, thereby ensuring physically meaningful dynamics. The min-max principle and compactness arguments yielded robust spectral stability bounds, while analytic perturbation theory enabled explicit first-order eigenvalue shifts in representative cases such as quartic

anharmonicity. Resolvent and semigroup identities quantified sensitivity of dynamics to perturbations and provided stability "certificates" that naturally inform reliable computation. Finally, an uncertainty-aware modeling layer based on intuitionistic fuzzy sets and fuzzy graph/hypergraph abstractions was proposed to encode uncertain perturbations in an interpretable, structured manner without abandoning rigorous operator stability constraints. Future work includes extending the analysis to multi-dimensional anisotropic oscillators, time dependent perturbations, non-self-adjoint effective operators in open quantum systems, and rigorous uncertainty propagation combining operator bounds with fuzzy/interval parameterizations.

## REFERENCES

1. N. Yogeesh and P. K. Chenniappan, "Operations on intuitionistic fuzzy directed graphs," *Journal of Advances and Scholarly Researches in Allied Education (JASRAE)*, vol. 3, no. 6, pp. 1–4, 2012.
2. N. Yogeesh and P. K. Chenniappan, "A conceptual discussion about an intuitionistic fuzzy-sets and its applications," *International Journal of Advanced Research in IT and Engineering*, vol. 1, no. 6, pp. 45–55, 2012.
3. N. Yogeesh and P. K. Chenniappan, "Study on intuitionistic fuzzy graphs and its applications in the field of real world," *International Journal of Advanced Research in Engineering and Applied Sciences*, vol. 2, no. 1, pp. 104–114, 2013.
4. N. Yogeesh and P. K. Chenniappan, "Illustrative study on intuitionistic fuzzy hyper-graphs and dual-intuitionistic fuzzy hyper-graphs," *International Journal of Engineering, Science and Mathematics*, vol. 2, no. 1, pp. 255–264, 2013.
5. N. Yogeesh and P. K. Chenniappan, "Study on hyper-graphs and directed hyper-graphs," *Journal of Advances and Scholarly Researches in Allied Education (JASRAE)*, vol. 5, no. 10, pp. 1–5, 2013.
6. N. Yogeesh, "Graphical representation of solutions to initial and boundary value problems of second order linear differential equation using FOOS (Free & Open Source Software)-Maxima," *International Research Journal of Management Science and Technology (IRJMST)*, vol. 5, no. 7, pp. 168–176, 2014.
7. Yogeesh, N "Solving linear system of equations with various examples by using Gauss method," *International Journal of Research and Analytical Reviews (IJRAR)*, vol. 2, no. 4, pp. 338–350, 2015.
8. N. Yogeesh, "A study of solving linear system of equations by Gauss-Jordan matrix method—An algorithmic approach," *Journal of Emerging Technologies and Innovative Research (JETIR)*, vol. 3, no. 5, pp. 314–321, 2016.
9. N. Yogeesh, "Theoretical framework of quantum perspectives on fuzzy mathematics: Unveiling neural mechanisms of consciousness and cognition," *NeuroQuantology*, vol. 15, no. 4, pp. 180–187, 2017, doi: 10.48047/nq.2017.15.4.1148.
10. T. Kato, *Perturbation Theory for Linear Operators*, 2nd ed. Berlin, Germany: Springer, 1976.
11. M. Reed and B. Simon, *Methods of Modern Mathematical Physics I: Functional Analysis*. New York, NY, USA: Academic Press, 1980.
12. M. Reed and B. Simon, *Methods of Modern Mathematical Physics IV: Analysis of Operators*. New York, NY, USA: Academic Press, 1978.
13. J. Weidmann, *Linear Operators in Hilbert Spaces*. New York, NY, USA: Springer, 1980.
14. G. Teschl, *Mathematical Methods in Quantum Mechanics: With Applications to Schrödinger Operators*, 2nd ed. Providence, RI, USA: AMS, 2014.
15. B. Simon, "The bound state of weakly coupled Schrödinger operators in one and two dimensions," *Annals of Physics*, vol. 97, no. 2, pp. 279–288, 1976.
16. C. M. Bender and T. T. Wu, "Anharmonic oscillator," *Physical Review*, vol. 184, pp. 1231–1260, 1969.
17. C. Cohen-Tannoudji, B. Diu, and F. Laloë, *Quantum Mechanics*, Vol. 1. New York, NY, USA: Wiley, 1977.
18. B. Helffer, *Semi-Classical Analysis for the Schrödinger Operator and Applications*. Berlin, Germany: Springer, 1988.
19. E. B. Davies, *Spectral Theory and Differential Operators*. Cambridge, U.K.: Cambridge University Press, 1995.
20. J. Glimm and A. Jaffe, *Quantum Physics: A Functional Integral Point of View*, 2nd ed. New York, NY, USA: Springer, 1987.
21. M. Shubin, *Pseudodifferential Operators and Spectral Theory*, 2nd ed. Berlin, Germany: Springer, 2001.
22. L. D. Landau and E. M. Lifshitz, *Quantum Mechanics: Non-Relativistic Theory*, 3rd ed. Oxford, U.K.: Pergamon Press, 1977.
23. J. J. Sakurai and J. Napolitano, *Modern Quantum Mechanics*, 2nd ed. Cambridge, U.K.: Cambridge University Press, 2017.

