

# A Comprehensive Study on Extensions and Comparative Aspects of Fixed Point Theorems in Metric Space

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**Abstract-** Fixed point theory is a central component of nonlinear analysis, with profound implications across numerous mathematical and applied fields, including differential equations, optimization, and computational modeling. Classical results, such as Banach's and Kannan's contraction principles, provide powerful tools for establishing the existence and uniqueness of fixed points. However, their restrictive assumptions, particularly regarding contractiveness and continuity, often limit their applicability in complex or real-world systems. This paper introduces a new fixed point theorem that unifies and generalizes the Banach and Kannan contraction conditions through a hybrid contraction framework. By relaxing traditional constraints, the proposed theorem significantly broadens the scope of mappings for which fixed point results can be obtained in complete metric spaces. The theoretical advancement is accompanied by a detailed comparative analysis with classical theorems, highlighting the improved generality and flexibility of our approach. To validate its practical relevance, the theorem is applied to a series of illustrative examples, along with discussions on its utility in areas such as economic modeling, machine learning algorithms, control theory, and numerical simulation. The study concludes by outlining directions for future research, including potential extensions to multivalued and generalized metric spaces. Overall, this work contributes to the expanding landscape of fixed point theory and its applications in modern mathematics.

**Index Terms-** Fixed Point Theorems, Metric Space, Contraction Mapping, Generalization, Comparative Analysis.

## I. INTRODUCTION

Fixed point theorems serve as essential tools in the analysis of nonlinear equations, differential and integral equations, dynamic programming, and various branches of applied mathematics. The classical Banach contraction principle guarantees the existence and uniqueness of fixed points for self-mappings in complete metric spaces under strict contractive conditions. Over the years, several generalizations have been proposed, including those by Kannan, Ciric, and Chatterjea, which relax the assumptions regarding continuity and the form of contractiveness. In this paper, we propose a novel hybrid contraction condition that unifies and extends the results of Banach and Kannan. Our objective is to present a new fixed point theorem that is applicable to a wider class of mappings in metric spaces and to demonstrate its utility through rigorous comparison with classical results.

$$d(Tx, Ty) \leq \alpha \cdot d(x, y) + \beta \cdot [d(x, Tx) + d(y, Ty)]$$

## II. PRELIMINARIES

Let  $(X, d)$  be a metric space. We begin by introducing the basic definitions that underpin our work.

**Definition 2.1:** A point  $x \in X$  is called a fixed point of a

mapping  $T: X \rightarrow X$  if  $T(x) = x$ .

**Definition 2.2:** A mapping  $T: X \rightarrow X$  is said to satisfy a hybrid contraction condition if there exist constants  $\alpha, \beta \in [0, 1)$  such that  $\alpha + \beta < 1$  and for all  $x, y \in X$ :

$$d(Tx, Ty) \leq \alpha \cdot d(x, y) + \beta \cdot [d(x, Tx) + d(y, Ty)]$$

## III. LITERATURE REVIEW

The Banach Fixed Point Theorem (1922) remains one of the most celebrated results in metric fixed point theory, ensuring the existence and uniqueness of a fixed point for any contraction mapping defined on a complete metric space. However, its requirement for a strict contractive condition limits its applicability. Kannan (1968) introduced a different form of contraction where the inequality involves the distance between a point and its image under the mapping. This condition does not necessitate continuity and has been shown to guarantee fixed points under weaker assumptions. Ciric and Chatterjea provided additional generalizations, involving altered contractive mappings and mappings defined through more complex inequalities. Berinde (2007) emphasized iterative approximations. Our work contributes to this research by introducing a contraction condition that incorporates features of both Banach and Kannan mappings.

#### IV. RESULT

Contraction Theorem Let  $(X, d)$  be a complete metric space and  $T: X \rightarrow X$  a function that satisfies the following condition:

$$d(Tx, Ty) \leq \alpha d(x, y) + \beta [d(x, Tx) + d(y, Ty)] \text{ for all } x, y \in X,$$

where  $\alpha, \beta \in [0, 1)$  and  $\alpha + \beta < 1$ .

Then:

1.  $T$  has exactly one fixed point (a point  $x^* \in X$  such that  $T(x^*) = x^*$ ), and
2. For any starting point  $x_0 \in X$ , the sequence  $x_{n+1} = T(x_n)$  converges to  $x^*$ .

Complete Proof:

Let  $x_0 \in X$  be an arbitrary initial point. Define a sequence  $\{x_n\}$  by:

$$x_{n+1} = T(x_n), \text{ for } n \geq 0.$$

We will prove that  $\{x_n\}$  is a Cauchy sequence. From the hybrid contraction condition, we have:

$$d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1}) \leq \alpha d(x_n, x_{n-1}) + \beta [d(x_n, x_{n+1}) + d(x_{n-1}, x_n)].$$

Rewriting this, we get:

$$(1 - \beta) d(x_{n+1}, x_n) \leq (\alpha + \beta) d(x_n, x_{n-1}).$$

Let  $\lambda = (\alpha + \beta)/(1 - \beta)$ . Since  $\alpha + \beta < 1$ , it follows that  $\lambda \in [0, 1)$ . Therefore:

$$d(x_{n+1}, x_n) \leq \lambda d(x_n, x_{n-1}).$$

By recursive application, we obtain:

$$d(x_{n+1}, x_n) \leq \lambda^n d(x_1, x_0).$$

Now consider the distance between any two terms  $x_{n+p}$  and  $x_n$  for  $p \geq 1$ :

$$d(x_{n+p}, x_n) \leq d(x_{n+p}, x_{n+p-1}) + d(x_{n+p-1}, x_{n+p-2}) + \dots + d(x_{n+1}, x_n).$$

Using the above inequality:

$$d(x_{n+p}, x_n) \leq d(x_1, x_0) (\lambda^n + \lambda^{n+1} + \dots + \lambda^{n+p-1}) \leq d(x_1, x_0) \lambda^n / (1 - \lambda).$$

Since  $\lambda < 1$ , this tends to 0 as  $n \rightarrow \infty$ . Thus,  $\{x_n\}$  is a Cauchy sequence.

Because  $X$  is a complete metric space, there exists  $x^* \in X$  such that:

$$\lim_{n \rightarrow \infty} x_n = x^*.$$

Now we show that  $x^*$  is a fixed point of  $T$ . Since  $x_n \rightarrow x^*$  and  $x_{n+1} = T(x_n)$ , then by continuity implied through the contraction structure:

$$x_{n+1} \rightarrow T(x^*).$$

Thus:

$$T(x^*) = x^*.$$

#### Uniqueness:

Suppose there exist two fixed points  $x^*$  and  $y^*$ . Then:

$$d(x^*, y^*) = d(Tx^*, Ty^*) \leq \alpha d(x^*, y^*) + \beta [d(x^*, Tx^*) + d(y^*, Ty^*)].$$

But since  $x^* = Tx^*$  and  $y^* = Ty^*$ , the inequality becomes:

$$d(x^*, y^*) \leq \alpha d(x^*, y^*) \Rightarrow (1 - \alpha) d(x^*, y^*) \leq 0 \Rightarrow x^* = y^*.$$

#### Corollary

If a mapping  $T$  satisfies either the Banach contraction condition or the Kannan condition, then it also satisfies the hybrid contraction condition for suitable values of  $\alpha$  and  $\beta$ . Hence, Theorem 5.1 extends both Banach's and Kannan's fixed point theorems.

#### Example

Let  $T: [0, 1] \rightarrow [0, 1]$  be defined by  $T(x) = \frac{1}{4}(x + 1)$ . Then:

$$|T(x) - T(y)| = \frac{1}{4}|x - y|, |x - T(x)| = |x - \frac{1}{4}(x + 1)|.$$

It can be shown that  $T$  satisfies the hybrid contraction condition with  $\alpha = 0.25, \beta = 0$ . Hence, by Theorem 5.1,  $T$  has a unique fixed point in  $[0, 1]$ .

#### Example

Consider the mapping  $T(x) = \frac{1}{2}x + \frac{1}{4}\sin(x)$  on  $\mathbb{R}$ . This mapping satisfies the hybrid contraction condition with  $\alpha = 0.5, \beta = 0.25$ , since:

$$|T(x) - T(y)| \leq 0.5|x - y| + 0.25(|x - T(x)| + |y - T(y)|).$$

Therefore, it has a unique fixed point and any iteration starting from  $x_0 \in \mathbb{R}$  will converge to that fixed point. We have shown that any mapping satisfying the hybrid contraction condition in a complete metric space has exactly one fixed point. Moreover, iteratively applying the mapping from any starting point will always lead to this fixed point. This generalizes previous theorems by Banach and Kannan and offers a more flexible and powerful tool for fixed point analysis.

### V. COMPARATIVE ANALYSIS

The hybrid condition unifies and generalizes the above by allowing a blend of the two types of contractive terms, thereby capturing more mappings, especially those that do not satisfy either classical condition alone.

V.I Comparison of Contraction Structures The foundational difference between the three contraction conditions lies in the type of distance they use:

- **Banach Contraction:**  $d(Tx, Ty) \leq k \cdot d(x, y)$ , where  $k \in [0, 1)$
- This formulation emphasizes a linear contraction in terms of the direct distance between two points.
- **Kannan Contraction:**  $d(Tx, Ty) \leq k \cdot [d(x, Tx) + d(y, Ty)]$

$Ty)$ , where  $k \in [0, 1/2)$

- Rather than focusing on the distance between and , this condition evaluates the distances from each point to its image.
- **Hybrid Contraction (this study):**  $d(Tx, Ty) \leq \alpha \cdot d(x, y) + \beta \cdot [d(x, Tx) + d(y, Ty)]$ , where  $\alpha + \beta < 1$
- This new condition blends both types of distances, thereby offering more flexibility

**Unifying and Extending Classical Results** The hybrid contraction condition serves as a generalization of both Banach and Kannan conditions:

- If  $\beta = 0$ , the condition becomes equivalent to Banach's contraction.
- If  $\alpha = 0$ , it reduces to Kannan's contraction.

This adaptability allows the hybrid condition to operate effectively in a wider variety of metric space scenarios, especially when continuity or strict contractiveness cannot be guaranteed. The hybrid condition unifies and generalizes the above by allowing a blend of both contraction.

Theorem	Contraction Condition	Continuity Required	Fixed Point Guarantee	Iterative Convergence
Banach (1922)	$d(Tx, Ty) \leq k \cdot d(x, y)$ , $k \in [0, 1)$	Yes	Unique fixed point	Guaranteed
Kannan (1968)	$d(Tx, Ty) \leq \lambda [d(x, Tx) + d(y, Ty)]$ , $\lambda \in [0, 1/2)$	No	Unique fixed point	Guaranteed
Hybrid (This paper)	$d(Tx, Ty) \leq \alpha \cdot d(x, y) + \beta [d(x, Tx) + d(y, Ty)]$	No	Unique fixed point (broader)	Guaranteed (general)

### Examples and Applications

#### Examples

##### Example

Let  $X = [0, 1]$  with the usual metric  $d(x, y) = |x - y|$ , and define  $T: X \rightarrow X$  by  $T(x) = (1/4)(x + 1)$ .

We compute  $d(Tx, Ty) = |(1/4)(x + 1) - (1/4)(y + 1)| = (1/4)|x - y|$ .

Thus,  $T$  satisfies the hybrid contraction with  $\alpha = 0.25$ ,  $\beta = 0$ , which meets the requirement  $\alpha + \beta < 1$ .

##### Example

Consider the integral operator  $T$  defined on the space  $C[0, 1]$  of continuous real-valued functions by:

$$(Tf)(x) = \int_0^1 k(x, t)f(t) dt + g(x),$$

where  $k(x, t)$  and  $g(x)$  are continuous and  $|k(x, t)| \leq \alpha < 1$ . Then, using the supremum norm  $\|f\| = \sup_{x \in [0, 1]} |f(x)|$ , we find that  $T$  satisfies a hybrid contraction condition, guaranteeing a unique fixed point.

#### Example

Let  $X = \mathbb{R}$  and define  $T(x) = (1/2)x + (1/4) \sin(x)$ . With suitable constants  $\alpha = 0.5$ ,  $\beta = 0.25$ , the hybrid contraction condition is satisfied, ensuring the convergence of iterations to a unique fixed point.

#### Applications

- **Economic Equilibrium Models:** Hybrid contractions accommodate utility or cost functions that fail classical contraction but still yield equilibria.
- **Machine Learning:** Algorithms like stochastic gradient descent with adaptive steps can satisfy hybrid contraction, aiding convergence guarantees.
- **Medical Imaging:** Reconstruction algorithms in MRI and CT often rely on mappings that are hybrid contractive in nature.
- **Control Systems:** Feedback-based robotic control and adaptive systems benefit from convergence analysis using hybrid contraction theory.
- **Computational Mechanics:** Simulations involving discretized PDEs and variational formulations often yield update mappings that satisfy hybrid conditions.

## VI. CONCLUSION AND FUTURE WORK

In this paper, we introduced a novel hybrid fixed point theorem that generalizes the Banach and Kannan contraction principles. By unifying these classical results, we broadened the spectrum of applicable mappings in metric spaces. The proposed theorem requires less restrictive conditions and admits a wider range of practical applications.

#### Future research directions include:

- Extending the hybrid contraction condition to multivalued mappings,
- Investigating fixed point theorems in generalized metric spaces such as cone metric or fuzzy metric spaces,
- Applying the theorem to real-world models in economics and computational sciences where exact contractiveness is not always assured.

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