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Common Fixed-Point Theorems Employing CLR Property in Sb-Metric Spaces

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Abstract- In this paper, we establish a fixed point theorem for four pairwise self-mappings that are weakly compatible in an Sb-metric Space. The results are obtained by combining a contractive condition with the CLR-property of pairs of mappings. Furthermore, we demonstrate that the CLR-property is a more general concept than the (E.A.)-property. First, the CLR-property is employed to establish the existence of common coincidence points, and subsequently, common fixed points are derived using the weak compatibility of the mappings. The results obtained in this study extend and generalize several well-known results available in the existing literature.

Keywords: Coincidence Point; Common Fixed Point; Sb-metric Space; CLR-property; Weak Compatibility.

I.INTRODUCTION AND PRELIMINARIES

The development of fixed point theory began with Brouwer, who established the fixed point theorem in finite-dimensional Euclidean spaces. In 1922, Birkhoff and Kellogg applied Brouwer's theorem to prove existence results for differential equations. Subsequently, Schauder generalized Brouwer's theorem to compact convex subsets of normed linear spaces, leading to the well-known Schauder Fixed Point Theorem. Later, Banach introduced the celebrated Fixed Point Theorem for contraction mappings, whose proof is simple and requires only a basic understanding of topology. Since then, numerous researchers have extended fixed point results to various generalized metric spaces, including b-metric, S-metric, and Sb-metric spaces. The concept of an Sb-metric space was introduced by Nizar Souayah et al. [1] and S. Radenović et al. [2]. This notion was developed by combining the concepts of S-metric spaces [5] and b-metric spaces [3,4]. Since its introduction, several researchers have established various fixed point theorems in the setting of Sb-metric spaces [11–15]. In the following, we present the definition of an S -metric space along with some of its fundamental topological properties.

Definition 1.1. [2] Let X be a non-empty set and $b \geq 1$ be a real number. An S_b-metric on X is a function S: $X^3 \to [0,\infty)$ that satisfy the following conditions, for each $u,v,w,a \in X$,
(S_b1) $S(u,v,w)=0$ if and only if $u=v=w$,
(S_b2) $S(u,v,w) \leq b[S(u,u,a)+S(v,v,a)+S(w,w,a)]$.
In this case, the pair (X, S) is called an S_b-metric space.
Since every S-metric is an S_b-metric with $b=1$, it is clear that S_b-metric spaces are the generalizations of S-metric spaces.
Example 1.2. [2] Let $X=R$ and let the function $S_b : X^3 \to [0,\infty)$ be defined as $S_b(u,v,w)=\{|u-w|+|v-w|\}^2$ is an S_b-metric on X with $b=4$.

Definition 1.3. [2] Let *(X, S)* is an S_b-metric space. A sequence $\{u_n\}$ in *X*

a) is said to converge to some $u \in X$, if to each $\varepsilon > 0$ there exists a number $n_0 \in N$ such that $S(u_n, u_n, u) < \in$ whenever $n > n_0$, that is, $S(u_n, u_n, u) \to 0$ as $n \to \infty$.

   a. Here, we write $u_n \to u$ as $n \to \infty$ or $\lim_{n\to\infty} u_n = u$.

b) is said to be a Cauchy sequence, if to each $\in > 0$ there exists a number $n_0 \in N$ such that $S(u_n, u_n, u_m) < \in$ whenever $m, n > n_0$, that is, $S(u_n, u_n, u_m) \to 0$ as $m, n \to \infty$.

The S_b-metric space *X* is said to be complete if every Cauchy sequence is convergent in *X*.

Lemma 1.4. [2] Let *(X,S)* is an S_b-metric space. Then for every $u, v, w \in X$,

(1) $S(u,u,v) \le bS(v,v,u)$

(2) $S(u,u,v) \le 2bS(u,u,w) + bS(v,v,w) \le 2bS(u,u,w) + b^2 S(w,w,v)$.

In 1986, Gerald Jungck [6] introduced the concept of compatibility as a generalization of the commutativity condition for mappings. Later, Jungck and Rhoades [7] proposed the notion of weak compatibility and showed that every pair of compatible mappings is weakly compatible, whereas the converse is not necessarily true. In 2002, Aamri and Moutawakil [8] introduced the property (E.A.), which has since become an important tool in the establishment of fixed point results. Subsequently, in 2011, Sintunavarat et al. [9] introduced the common limit range (CLR) property. Using this property, they demonstrated that the closedness of the range of any of the involved mappings is not a necessary condition for the existence of fixed points.

Definition 1.5. Let *(X, S)* be an S_b-metric space and *F, G* be two self-maps of *X*. Then the pair *(F, G)* is

(i) said to be compatible **[6]** if $\lim_{n\to\infty} S(FGu_n, FGu_n, GFu_n) = 0$ for every sequence $\{u_n\}$ in *X* such that $\lim_{n\to\infty} Fu_n = \lim_{n\to\infty} Gu_n = t \in X$

(ii) said to be weakly compatible **[7]** if $FGu = GFu$ for every $u \in X$ such that $Fu = Gu$.

(iii) said to satisfy (E.A) property [**8**] if there exists a sequence $\{u_n\}$ in X such that $\lim_{n\to\infty} Pu_n = \lim_{n\to\infty} Qu_n = t, t \in X$.

Definiton 1.6. (Sintunavarat and Kumam, [**9**]) The pair *(F,G)* of self-mappings of *X*, where *X* is an S_b-metric space is said to satisfy the common limit in the range of *G* *(*CLR_G*)*- property if there exists a sequence $\{u_n\}$ in *X* such that $\lim_{n\to\infty} Fu_n = \lim_{n\to\infty} Gu_n = Gu$, $u \in X$.

Definiton 1.7. (Chauhan, [**10**]) Two pairs *(F,P)* and *(G,Q)* of self-mappings of *X*, where *X* is an S_b-metric space are said to satisfy the common limit in the common range of both *P* and *Q* *(*CLR_{PQ}*)*- property if there exists sequences $\{u_n\}$ and $\{v_n\}$ in *X* such that $\lim_{n\to\infty} Fu_n = \lim_{n\to\infty} Pu_n = \lim_{n\to\infty} Gv_n = \lim_{n\to\infty} Qv_n = p$, where $p \in P(X) \cap Q(X)$.

Definition 1.8. Let *X* be a non-empty set and *F, G* be two self-maps of *X*. Then a point $u \in X$ is said to be a coincidence point of *F* and *G*, if $Fu = Gu$. We denote $C(F,G) = \{u \in X : Fu = Gu\}$.

The present study investigates and establishes fixed point theorems involving four pairwise weakly compatible self-maps in the setting of a complete S_b-metric space.

II. MAIN RESULTS

Theorem 2.1: *Let A,B,P and Q be self-maps of an S_b-metric space (X,S) with* $b>1$ *such that* $b^k S(Au,Au,Bv)\le M_b(u,v)$ *for all* $u,v\in X$ *, where k > 1 is a constant and*

$$M_b(u,v)=\max\left\{S(Pu,Pu,Qv),S(Au,Au,Pu),S(Bv,Bv,Qv),\frac{1}{2b}\left[S(Au,Au,Qv)+S(Pu,Pu,Bv)\right]\right\}. \qquad (2.1)$$

if the pairs (A, P) and (B,Q) satisfy (CLR_{PQ})-property, then $C(A,P)\ne\phi$ *and* $C(B,Q)\ne\phi$*.*

Furthermore, if the pairs (A, P) and (B,Q) are weakly compatible then the maps A, B, P and Q have a unique common fixed point.

Proof : Since *the pairs (A, P) and (B,Q) satisfy (CLR_{PQ})-property,* there must be a sequences $\{u_n\}$ and $\{v_n\}$ in *X* such that $\lim_{n\to\infty}Au_n=\lim_{n\to\infty}Pu_n=\lim_{n\to\infty}Bv_n=\lim_{n\to\infty}Qv_n=p,$ where $p\in P(X)\cap Q(X).$

(2.2)

Then $p=\Pr=Qt,$ for some $r,t\in X.$ (2.3)

We now prove that $p=Ar$

By replacing *u, v* with r,v_n respectively in (2.1), we get

$$b^k S(Ar,Ar,Bv_n)\le M_b(r,v_n)$$

where, $M_b(r,v_n)=\max\left\{S(\Pr,\Pr,Qv_n),S(Ar,Ar,\Pr),S(Bv_n,Bv_n,Qv_n),\frac{1}{2b}\left[S(Ar,Ar,Qv_n)+S(\Pr,\Pr,Bv_n)\right]\right\}$

$$=\max\left\{S(p,p,Qv_n),S(Ar,Ar,p),S(Bv_n,Bv_n,Qv_n),\frac{1}{2b}\left[S(Ar,Ar,Qv_n)+S(\Pr,\Pr,Bv_n)\right]\right\}$$

On taking limit as $n\to\infty,$ and using (2.3), we get

$$\lim_{n\to\infty}b^k S(Ar,Ar,Bv_n)\le\lim_{n\to\infty}M_b(r,v_n)$$

$$=\lim_{n\to\infty}\max\left\{S(p,p,Qv_n),S(Ar,Ar,p),S(Bv_n,Bv_n,Qv_n),\frac{1}{2b}\left[S(Ar,Ar,Qv_n)+S(\Pr,\Pr,Bv_n)\right]\right\}$$

$$=\max\left\{S(p,p,p),S(Ar,Ar,p),S(p,p,p),\frac{1}{2b}\left[S(Ar,Ar,p)+S(p,p,p)\right]\right\}$$

$$=\max\left\{S(Ar,Ar,p),\frac{1}{2b}\left[S(Ar,Ar,p)\right]\right\}$$

$$=S(Ar,Ar,p)\text{ since }b>1.$$

Thus $b^k S(Ar,Ar,p)\le S(Ar,Ar,p).$

This implies $S(Ar,Ar,p)=0,$ because $b^k\ge b>1.$

Thus $Ar=p.$ (2.4)

From (2.3) and (2.4), $p=Ar=\Pr$ and hence $C(A,P)\ne\phi.$ (2.5)

We now prove that $p=Bt.$

By taking $u=r,v=t$ in (2.1), we get

$$b^k S(Ar,Ar,Bt)\le M_b(r,t),$$

$$M_b(r,t)=\max\left\{S(\Pr,\Pr,Qt),S(Ar,Ar,\Pr),S(Bt,Bt,Qt),\frac{1}{2b}\left[S(Ar,Ar,Qt)+S(\Pr,\Pr,Bt)\right]\right\}$$

$$=\max\left\{S(p,p,p),S(p,p,p),S(Bt,Bt,p),\frac{1}{2b}\left[S(p,p,p)+S(p,p,Bt)\right]\right\}$$

$$=\max\left\{S(Bt,Bt,p),\frac{1}{2b}\left[S(p,p,Bt)\right]\right\}$$

$$=S(Bt,Bt,p)\text{ since }b>1.$$

Thus $b^k S(p,p,Bt)=b^k S(Ar,Ar,Bt)\le S(p,p,Bt),$

This implies $S(p,p,Bt)=0,$ because $b^k \geq b > 1.$

Thus $Bt = p.$

(2.6)

Therefore, from (2.3) and (2.6), $p = Bt = Qt$ and hence $C\left(B,Q\right) \neq \phi.$ (2.7)

From (2.5) and (2.7), $p = Ar = \Pr = Bt = Qt.$ (2.8)

From the weak compatible property of the pairs *(A,P)* and *(B,Q)*, it follows that

Ap = *Pp* and *Bp* = *Qp*. (2.9)

Now, we will show that $Ap = p.$

From (2.1), we have $b^k S(Ap,Ap,p) = b^k S(Ap,Ap,Bt) \leq M_b(p,t),$

where $M_b(p,t) = \max\left\{S(Pp,Pp,Qt), S(Ap,Ap,Pp), S(Bt,Bt,Qt), \dfrac{S(Ap,Ap,Qt)+S(Pp,Pp,Bt)}{2b}\right\}$

$$= \max\left\{S(Ap,Ap,p), S(Ap,Ap,Ap), S(p,p,p), \frac{S(Ap,Ap,p)+S(Ap,Ap,p)}{2b}\right\},$$ by (2.8), (2.9)

$$= \max\left\{S(Ap,Ap,p), \frac{1}{b}S(Ap,Ap,p)\right\},$$

$$= S(Ap,Ap,p) \text{ since } b > 1.$$

Hence $b^k S(Ap,Ap,p) \leq S(Ap,Ap,p),$

This follows $p = Ap = Pp,$ because $b^k \geq b > 1.$

Similarly, from (2.1), we have $b^k S(p,p,Bp) = b^k S(Ap,Ap,Bp) \leq M_b(p,p),$

where $M_b(p,p) = \max\left\{S(Pp,Pp,Qp), S(Ap,Ap,Pp), S(Bp,Bp,Qp), \dfrac{S(Ap,Ap,Qp)+S(Pp,Pp,Bp)}{2b}\right\}$

$$= \max\left\{S(p,p,Bp), S(p,p,p), S(Bp,Bp,Bp), \frac{S(p,p,Bp)+S(p,p,Bp)}{2b}\right\}$$

$$= \max\left\{S(p,p,Bp), \frac{1}{b}S(p,p,Bp)\right\},$$

$$= S(p,p,Bp),$$

Hence $b^k S(p,p,Bp) \leq S(p,p,Bp),$

This follows $p = Bp = Qp,$ because $b^k \geq b > 1.$

Therefore $Ap = Pp = p = Bp = Qp.$

To prove uniqueness of *p*, if possible suppose that p^* $(p \neq p^*)$ be another common fixed of *A*,*B*,*P* and *Q* . Then $Ap^* = Pp^* = p^* = Bp^* = Qp^*.$

From (2.1), $b^k S(p,p,p^*) = b^k S(Ap,Ap,Bp^*) \leq M_b(p,p^*)$

and $M_b(p,p^*) = \max\left\{S(Pp,Pp,Qp^*), S(Ap,Ap,Pp), S(Bp^*,Bp^*,Qp^*), \dfrac{S(Ap,Ap,Qp^*)+S(Pp,Pp,Bp^*)}{2b}\right\}$

$$= \max\left\{S(p,p,p^*), S(p,p,p), S(p^*,p^*,p^*), \frac{S(p,p,p^*)+S(p,p,p^*)}{2b}\right\}$$

$$= \max\left\{S(p,p,p^*), \frac{1}{b}S(p,p,p^*)\right\},$$

$= S(p,p,p^*),$ since $b > 1.$

Hence, $b^k S(p,p,p^*) \leq S(p,p,p^*),$

Since $b^k \geq b > 1,$ this is a contradiction to our supposition $p \neq p^*.$

Therefore $p = p^*.$

III. CONCLUSION

In this study, common fixed point theorems for four weakly compatible self-maps in an Sb-metric space are established under suitable contractive conditions. By utilizing the Common Limit Range (CLR) property, which offers a more general framework than conventional assumptions, the existence and uniqueness of a common fixed point are obtained. Several related results and corollaries are also derived, including special cases involving only two self-maps. These results contribute to the advancement of Fixed Point Theory in generalized metric spaces by extending and unifying a number of classical fixed point theorems from metric and related spaces within the broader setting of Sb-metric spaces.

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