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Coefficient Inequalities for a Subclass of Meromorphically Star Like Functions

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Abstract- The object of the present paper is to introduce new subclasses of Meromorphically star like functions and to obtain certain coefficient inequalities for the functions in these classes. The Fekete–Szegö inequality for the functions in these classes is also obtained.

Keywords: Analytic function, Meromorphically, star like functions, Fekete–Szegö inequality. 2000 Mathematics subject classification: 30 c 45.

I.INTRODUCTION

Let 'Σ' be the class of functions of the form

$$f(z)=\frac{1}{z}+\sum_{n=0}^{\infty}a_n z^n \qquad (1.1)$$

Which are regular in $D=\{z;0<|z|<1\}$with a simple pole at the origin.
Let $\sum_s$, $\sum^*(\alpha)$ and $\sum_k(\alpha)$ $(0\le\alpha<1)$ denote the subclasses of 'Σ' that are univalent, meromorphically star like of order' α 'and meromorphically convex of order' α 'respectively.

Analytically a function 'f' of the form (1.1) is in $\sum^*(\alpha)$ if

$$\operatorname{Re}\left\{-\frac{zf'(z)}{f(z)}\right\}>\alpha \qquad (0\le\alpha<1) \quad \forall\ z\in D$$

Also a function 'f' of the form (1.1) is in $\sum_k(\alpha)$ iff

$$\operatorname{Re}\left\{-\left(1+\frac{zf''(z)}{f'(z)}\right)\right\}>\alpha \qquad (0\le\alpha<1) \quad \forall\ z\in D$$

These classes have been extensively studied by Pomuranke[3], Cluni[1], Millar[2], Royster [4], and others. In this section we introduce a subclass of meromorphically star like function; we obtain the coefficient inequalities and FeketeSzego inequalities for the functions in this class. To prove our results we require following lemma.

Lemma(1.1)[5] If $p(z)=1+c_1z+c_2z^2+c_3z^3+\dots$ is a function with positive real part and $p(0)=1$ then for any complex number v, we have

$$|c_2-vc_1^2|\le 2\max\{1,|2v-1|\}$$

This result is sharp for the functions

$$p(z)=\frac{1+z^2}{1-z^2} \text{ and } p(z)=\frac{1+z}{1-z}$$

We now define the class $\sum^{*}(\alpha,\beta)$ as follows

Definition (1.1): A function $f \in \sum$ of the form(1.1)is said to be in the class $\sum^{*}(\alpha,\beta)$ if

$$\operatorname{Re}\left\{-\frac{zf'(z)}{f(z)}\right\} > \alpha\left|\frac{zf'(z)}{f(z)}+1\right| + \beta \qquad \forall\ z \in D \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(1.2)}$$

$$\text{For} \qquad \alpha \geq 0 \quad , \quad 0 \leq \beta < 1$$

The class of all such functions is denoted by $\sum^{*}(\alpha,\beta)$.

II.COEFFICIENT INEQUALITY

In this section, we establish sufficient conditions for a function f to belong to the class $\sum^{*}(\alpha,\beta)$ We also derive coefficient inequalities and the Fekete–Szegö inequality for functions f in this class $\sum^{*}(\alpha,\beta)$.

Theorem2.1: If $f(z) \in \sum^{*}(\alpha,\beta)$ with $0 \leq \alpha \leq \beta$ then $f(z) \in \sum^{*}\left(\frac{\beta-\alpha}{1-\alpha}\right)$

Proof: Let $f(z) \in \sum^{*}(\alpha,\beta)$

$$\Rightarrow \operatorname{Re}\left(-\frac{zf'(z)}{f(z)}\right) > \alpha\left|\frac{zf'(z)}{f(z)}+1\right| + \beta$$

$$\Rightarrow \operatorname{Re}\left(-\frac{zf'(z)}{f(z)}\right) > \alpha\left|-\frac{zf'(z)}{f(z)}-1\right| + \beta$$

$(\therefore R(w) \leq |w|$ For any complex number w)

$$\Rightarrow \operatorname{Re}\left(-\frac{zf'(z)}{f(z)}\right) > \alpha \operatorname{Re}\left(-\frac{zf'(z)}{f(z)}-1\right) + \beta$$

$$\Rightarrow \operatorname{Re}\left(-\frac{zf'(z)}{f(z)}\right) > \frac{\beta-\alpha}{1-a}\ (z \in U)$$

$$\Rightarrow f(z) \in \sum^{*}\left(\frac{\beta-\alpha}{1-\alpha}\right) \qquad (2.1)$$

Here if $0 \leq \alpha \leq \beta$ then we have

$$0 \leq \frac{\beta-\alpha}{1-\alpha} < 1$$

Theorem 2.2

If $f(z) \in \sum^{*}(\alpha,\beta)$ then

$$|a_0| \leq \frac{2(1-\alpha)}{1-\beta} \qquad (2.2)$$

and
$$|a_n| \leq \frac{2(1-\alpha)}{(1-\beta)(n+1)} \mathop{\pi}_{j=1}^{n}\left(1+\frac{2(1-\alpha)}{j(1-\beta)}\right) n \geq 1 \qquad (2.3)$$

Proof: If $f(z) \in \sum^{*}(\alpha,\beta)$ then from theorem (2.1)

$$\operatorname{Re}\left(\frac{-zf'(z)}{f(z)}\right) > \frac{\beta-\alpha}{1-\alpha}, \qquad z \in U$$

Define the function $p(z)$ by

$$p(z) = \frac{(1-\alpha)\left(\frac{-zf^{|}(z)}{f(z)}\right) - (\beta-\alpha)}{1-\beta} = 1 + \sum_{k=1}^{\alpha} p_k z^k \;\; \forall \; z \in U$$

Here $p(z)$ is analytic in U with $p(0) = 1$ and $\operatorname{Re}\, p(z) > 0$

$$\Rightarrow 1 + \sum_{k=1}^{\alpha} p_k z^k = \frac{(1-\alpha)\left(\frac{-zf^{|}(z)}{f(z)}\right) - (\beta-\alpha)}{1-\beta}$$

$$(1-\beta)\left[1 + \sum_{k=1}^{\alpha} p_k z^k\right] = (1-\alpha)\left(\frac{-zf^{|}(z)}{f(z)}\right) - (\beta-\alpha)$$

$$(1-\beta)\left(1 + \sum p_k z^k\right)\left(\frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n\right) = (1-\alpha)\left(-zf^{|}(z)\right) - (\beta-\alpha)\, f(z)$$

$$(1-\beta)\left(\frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n + \sum_{k-1}^{\infty} p_k z^{k-1} + \left(\sum_{k-1}^{\infty} p_k z^k\right)\left(\sum_{n=0}^{\infty} a_n z^n\right)\right) =$$

$$(1-\alpha)\left(\frac{1}{z} - \sum_{n=0}^{\infty} n a_n z^n\right) - (\beta-\alpha)\left(\frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n\right)$$

Comparing the coeff.of z^n

$$(1-\beta)\left[a_n + p_{n+1} + p_1 a_{n-1} + p_2 a_{n-2} + \ldots + p_{n-2} a_2 + p_{n-1} a_1 + p_n a_0\right]$$

$$= \sum a_n \left[-n(1-\alpha) - (\beta-\alpha)\right]$$

$$a_n\left[-n + n\alpha - \beta + \alpha - 1 + \beta\right] = (1-\beta)\left[p_{n-1} + a_n p_n + a_1 p_{n-1} + \ldots + a_{n-2} p_2 + a_{n-1} p_1\right]$$

$$-\, a_n (1-\alpha)(n+1) = (1-\beta)\left[p_{n+1} + a_0 p_n + a_1 p_{n-1} + \ldots + a_{n-2} p_2 + a_{n-1} p_1\right]$$

$$a_n = \frac{(1-\beta)}{-(1-\alpha)(n+1)}\left[p_{n+1} + a_0 p_n + a_1 p_{n-1} + \ldots + a_{n-2} p_2 + a_{n-1} p_1\right]$$

$$|a_n| \le \frac{2(1-\beta)}{(1-\alpha)(n+1)}\left[1 + |a_0| + |a_1| + \ldots + |a_{n-2}| + |a_{n-1}|\right]$$

If $n = 0$ then

$$|a_0| \le \frac{2(1-\beta)}{(1-\alpha)}$$

Thus the coefficient estimate (2.2) holds true for $n = 0$

If $n = 1$

$$|a_1| \le \frac{1-\beta}{1-\alpha}\left[1 + |a_0|\right]$$

$$\le \frac{1-\beta}{1-\alpha}\left[1 + \frac{2(1-\beta)}{1-\alpha}\right]$$

Thus the inequality (2.3) is true for $n = 1$

Assume the result (2.3) is true for $n = k$

i.e. $$|a_k| \le \frac{2(1-\beta)}{(1-\alpha)(k+1)} \overset{k}{\underset{j=1}{\pi}} \left(1 + \frac{2(1-\beta)}{j(1-\alpha)}\right)$$

For $n = k+1$

Consider

$$|a_{k+1}| \leq \frac{2(1-\beta)}{(1-\alpha)(k+2)}\left[1+\frac{2(1-\beta)}{1-\alpha}+\frac{2(1-\beta)}{2(1-\alpha)}\left(1+\frac{2(1-\beta)}{(1-\alpha)}\right)\right.$$

$$\left.+\ldots+\frac{2(1-\beta)}{(1-\alpha)(k+1)}\overset{k}{\underset{j=1}{\pi}}\left[1+\frac{2(1-\beta)}{j(1-\alpha)}\right]\right]$$

$$|a_{k+1}| \leq \frac{2(1-\beta)}{(1-\alpha)(k+2)}\overset{k+1}{\underset{j=1}{\pi}}\left[1+\frac{2(1-\beta)}{j(1-\alpha)}\right]$$

The result is true for $n = k+1$.

$\therefore$ The result (2.3) is true for all $n \geq 1$

By mathematical induction, this completes the proof of the theorem (2.2)

III. FEKETESZEGO INEQUALITY

Theorem 3.1: If $f(z) \in \sum{}^*(\alpha, \beta)$ then for any complex number μ we have

$$\left|a_1 - \mu a_0^2\right| \leq \frac{1-\beta}{2(1-\alpha)}\left[2 \max\left(1, \left|\left(\frac{1-\beta}{1-\alpha}\right)(2-4\mu)-1\right|\right)\right]$$ and the result is sharp

Proof: Since $f(z) \in \sum{}^*(\alpha, \beta)$ from theorem (2.2) we have

$$a_0 = \frac{(1-\beta)}{-(1-\alpha)} p_1$$

$$a_1 = \frac{-(1-\beta)}{(1-\alpha).2}\left[p_2 + a_0 p_1\right]$$

$$a_1 = \frac{-(1-\beta)}{2(1-\alpha)} \cdot p_2 + \frac{(1-\beta)^2}{2(1-\alpha)^2} \cdot p_1^2$$

For any complex number μ.

Consider

$$a_1 - \mu a_0^2 = \frac{-(1-\beta)}{2(1-\alpha)} \cdot p_2 + \frac{(1-\beta)^2}{2(1-\alpha)^2} p_1^2 - \mu\frac{(1-\beta)^2}{(1-\alpha)^2} \cdot p_1^2$$

$$= \frac{-(1-\beta)}{2(1-\alpha)}\left[p_2 - \frac{2(1-\alpha)}{(1-\beta)}\left[\frac{(1-\beta)^2}{2(1-\alpha)^2} - \mu\frac{(1-\beta)^2}{(1-\alpha)^2}\right] p_1^2\right]$$

$$= \frac{-(1-\beta)}{2(1-\alpha)}\left[p_2 - \left[\frac{(1-\beta)}{(1-\alpha)} - 2\mu\frac{(1-\beta)}{(1-\alpha)}\right] p_1^2\right]$$

$$\left|a_1 - \mu a_0^2\right| = \frac{1-\beta}{2(1-\alpha)}\left|p_2 - \left[\frac{(1-\beta)}{(1-\alpha)} - 2\mu\frac{(1-\beta)}{(1-\alpha)}\right] p_1^2\right|$$

Where $$V = \frac{1-\beta}{1-\alpha}\left[1-2\mu\right]$$

By applying the lemma (1.1) we have

$$\left|p_2 - Vp_1^2\right| \leq 2 \max\left(1, \left|2\left(\frac{1-\beta}{1-\alpha}\right)(1-2\mu)-1\right|\right)$$

$$\leq 2 \max\left(1, \left|\left(\frac{1-\beta}{1-\alpha}\right)(2-4\mu)-1\right|\right)$$

$$\therefore \left|a_1 - \mu a_0^2\right| \le \frac{1-\beta}{2(1-\alpha)} \left|2\max\left(1, \left|\frac{(1-\beta)}{(1-\alpha)}(2-4\mu)-1\right|\right)\right|$$

This completes the proof of the theorem (2.3).
The result is sharp i.e.

$$\left|a_1 - \mu a_0^2\right| = \frac{1-\beta}{1-\alpha} \text{ if } p(z) = \frac{1+z^2}{1-z^2} \text{ and}$$

$$\left|a_1 - \mu a_0^2\right| = \frac{1-\beta}{1-\alpha}\left|2\left(\frac{1-\beta}{1-\alpha}\right)(1-2\mu)-1\right| \text{ if } p(z) = \frac{1+z}{1-z}$$

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