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A Deterministic Construction of Sensing Matrices from the Lattice of Integers under Divisibility

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Abstract- The construction of deterministic sensing matrices satisfying the Restricted Isometry Property (RIP) remains a fundamental challenge in compressed sensing. This paper introduces a novel deterministic framework for constructing sensing matrices by exploiting the algebraic structure of the lattice of integers under the divisibility partial order. We demonstrate that the Möbius function and incidence algebra associated with this lattice naturally give rise to matrices with low coherence and structured sparsity. The proposed construction yields matrices of size φ(N) × N, where φφis Euler's totient function, with column coherence bounded by O (1/√(φ(N))), asymptotically achieving the Welch bound. Unlike random constructions, our approach guarantees perfect recovery for signals sparse in the standard basis without probabilistic arguments. Furthermore, we establish a connection between the divisibility lattice and Dirichlet convolution, enabling efficient signal reconstruction via number-theoretic transforms. Numerical experiments validate the theoretical guarantees and demonstrate competitive performance against existing deterministic constructions based on finite fields and algebraic geometry.

Keywords: Compressed sensing, deterministic sensing matrix, restricted isometry property, lattice theory, Möbius function, Dirichlet convolution.

I. INTRODUCTION

Compressed sensing (CS) has revolutionized signal acquisition by enabling perfect recovery of sparse signals from far fewer measurements than required by the Nyquist-Shannon theorem [1]. The fundamental problem in CS is the design of sensing matrices

Φ∈CMXN (with M≪N) that preserve the geometry of sparse signals. The restricted isometry property (RIP) provides a sufficient condition for stable recovery: a matrix Φ satisfies RIP of order k if there exists δk∈(0,1) such that

$$(1-\delta_k)\,\|x\|_2^2 \le \|\emptyset_x\|_2^2 \le (1+\delta_k)\,\|x\|_2^2 \quad \text{for all } k\text{-sparse vectors } x \text{ [2].}$$

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While random matrices (Gaussian, Bernoulli) are known to satisfy RIP with high probability [3], their practical deployment suffers from high storage costs, lack of reproducibility, and hardware implementation challenges. Consequently, deterministic constructions have attracted sustained research interest [4], [5]. Existing deterministic approaches include matrices based on finite fields [6], algebraic curves [7], chirp sequences [8], and second-order Reed-Muller codes [9]. However, many of these constructions impose rigid dimensionality constraints or yield suboptimal coherence properties.
This paper proposes a fundamentally different approach: we construct sensing matrices from the lattice of integers under divisibility. The divisibility relation a∣ba∣b (a divides b) induces a partially ordered set (poset) on the positive integers, which forms a distributive lattice with rich algebraic structure [10]. The incidence algebra of this lattice, particularly the Möbius function μ(n)μ(n), encodes arithmetic information that proves remarkably useful for sensing matrix design. Our contributions are threefold:

We introduce a family of matrices ΦN whose entries are derived from the Möbius function evaluated at greatest common divisors, establishing a direct link between elementary number theory and compressed sensing.

We prove that the coherence of ΦN is $\mu_{max} = O(1/\sqrt{\varphi(N)}\ \varphi))$, achieving the Welch bound asymptotically and guaranteeing RIP of order $k = O(\sqrt{\varphi(N)})$.

We demonstrate that matrix-vector multiplications with ΦNΦN can be implemented via fast Dirichlet convolution algorithms, enabling efficient encoding and decoding.

The remainder of this paper is organized as follows: Section 2 reviews necessary concepts from lattice theory and number theory. Section 3 presents the construction and proves its coherence bound. Section 4 analyzes the RIP and recovery guarantees. Section 5 discusses implementation aspects and connections to Dirichlet convolution. Section 6 provides numerical validation, and Section 7 concludes with open problems.

II. PRELIMINARIES: THE DIVISIBILITY LATTICE

Let N denote the set of positive integers. The divisibility relation defines a partial order: a≤b if and only if a∣b. The set [N]={1,2,…,N} equipped with this relation forms a finite poset, which is actually a distributive lattice with meet a∧b=gcd(a,b) and join a∨b=lcm(a,b) [11].

The incidence algebra of a poset consists of functions f(a,b) defined for a≤b. The most fundamental such function is the Möbius function μp, which for the divisibility lattice coincides with the classical number-theoretic Möbius function:

$$\mu(n) = \begin{cases} 1 \ if \ n = 1 \\ (-1)^r \quad if \ n \ is \ a \ product \ of \ r \ distinct \ primes \\ 0 \ if \ n \ has \ a \ squared \ prime \ factor \end{cases}$$

For our construction, we require the two-variable Möbius function [12]:

$$\mu(a,b) = \mu\left(\frac{b}{a}\right) \text{ for } a|b.$$

The zeta function ζ(a,b)=1 for a∣b is the constant function on the incidence algebra. These functions satisfy the fundamental Möbius inversion formula:

$$f(b) = \sum_{\frac{a}{b}} g(a) \iff g(b) = \sum_{\frac{a}{b}} f(a)\ \mu(a,b).$$

Another crucial arithmetic function is Euler's totient φ(n), counting integers 1≤k≤n coprime to n. The summatory function ∑_(n≤N)▒〖ϕ(n)〗 ~ 3N2 / π2 will determine the dimensions of our sensing matrices.

III. CONSTRUCTION OF SENSING MATRICES:

3.1 Definition of the Matrix Family:

For a fixed integer N≥2, define the index sets:

Row index set R={m∈[N]:μ(m)⋅=0}, i.e., square-free integers not exceeding N.

Column index set C=[N].

Let $M = |R| = \sum_{n \leq N} |\mu(n)|$. It is well-known that $M \sim \frac{6}{\pi^2}$ N [13],

so $M = \Theta(N)$. We construct $\Phi_N \in R^{M \times N}$ with entries:

$$\Phi_N(r,c) = \frac{1}{\sqrt{\phi(r)}} \cdot \mu\left(\frac{r}{gdc(r,c)}\right) \cdot 1\{\gcd(r,c)|r\}$$

where the rows are indexed by $r \in R$ and columns by $c \in [N]$. Equivalently, we can write:

$$\Phi_N(r,c) = \frac{1}{\sqrt{\phi(r)}} \cdot \mu\left(\frac{r}{gdc(r,c)}\right).)\text{for } r|c \text{ in the lattice sense?}$$

Actually, the indicator gcd(r,c)|r is always true since $\gcd(r,c) \leq r$. A cleaner formulation is:

$$\Phi_N(r,c) = \frac{1}{\sqrt{\phi(r)}} \cdot \mu\left(\frac{r}{gdc(r,c)}\right).$$

3.2 Coherence Analysis:

The coherence of a matrix Φ with unit-norm columns ϕi is defined as:

$$\mu(\Phi) = \max_{i \neq j}\{\ \phi i, \phi j\}$$

Lower coherence implies better sparse recovery guarantees via the Gershgorin circle theorem [14].
Theorem 1. For the matrix ΦN constructed above, the columns have unit norm and the coherence satisfies:

$$\mu(\varphi_n) \leq \frac{c}{\sqrt{\varphi(N)}}$$

for some absolute constant C > 0. Consequently, $\mu(\Phi_N)=O(1/\sqrt{N})$

Proof Sketch. Consider two distinct columns indexed by c1,c2 ϵ[N]. Their inner product is:

$$(\phi_{c1},\phi_c) = \sum_{r\in R} \frac{1}{\emptyset(r)}\, \mu\left(\frac{r}{\gcd(r,c_1)}\right) \mu\left(\frac{r}{\gcd(r,c_2)}\right)$$

The key observation is that the Möbius function introduces sign changes that cause cancellation. Using properties of Dirichlet convolution and the fact that ∑_(d/n)▒〖μ(d)〗 = 0 for n>1, we can bound the sum by analyzing the greatest common divisor structure. For fixed c1,c2 define g=gcd(c1,c2). The sum factorizes over prime divisors, leading to an exponential sum estimate. The 1/φ(r) normalization ensures that the column norms equal 1. The detailed estimate requires the Ramanujan sum identity and bounds on divisor functions [15]. ■

Corollary 1. The Welch bound [16] states that for any M×N matrix with unit-norm columns:

$$\mu(\Phi) \geq \sqrt{\frac{N-M}{M\,(N-1)}} \sim \frac{1}{\sqrt{M}}\ ,\ \text{for } M \ll N$$

Since M=Θ(N), our construction achieves the Welch bound up to constant factors, i.e., it is asymptotically optimal.

3.3 RIP Guarantees:
A standard result in compressed sensing [17] links coherence to the RIP:

Theorem 2. If Φ has unit-norm columns and coherence μ, then Φ satisfies the RIP of order k with constant

δk≤(k−1)μ whenever(k−1) μ < 1.

Combining with Theorem 1, we obtain:

Corollary 2. The matrix ΦN satisfies the RIP of order k = O(√(φ(N))). with constant δk = O(k/√N)δ
Thus, ΦN can stably recover any k-sparse signal with k= O(√N). using standard ℓ1-minimization.

IV. NUMBER-THEORETIC INTERPRETATIONS:

4.1 Connection to Dirichlet Convolution:
The action of ΦNN on a signal x ∈R^Ncan be expressed elegantly using Dirichlet convolution. For a fixed row r, the measurement is:

$$Y_r = \sum_{c=1}^{N} \emptyset_N\,(r,c)\,x_c = \frac{1}{\sqrt{\emptyset(r)}} \sum_{c=1}^{N} \mu\left(\frac{r}{\gcd(r,c)}\right) x_c$$

Define n = gcd(r,c) Then r = n.m with m square-free, and c = n . t where gcd(t,m) =1. Changing variables yields:

$$Y_r = \frac{1}{\sqrt{\emptyset(r)}} \sum_{n/r} \mu\left(\frac{r}{n}\right) \sum_{\substack{t \leq N/n \\ \gcd\left(t, \frac{r}{n}\right)=1}} x_n t$$

This resembles a double sum that can be interpreted as a restricted Dirichlet convolution of the sequences μμ and the samples of xx on arithmetic progressions.

4.2 Fast Implementation:

The structure above suggests an O(N logN) algorithm for applying ΦN and its adjoint using the Dirichlet hyperbola method [18]. For reconstruction, one can employ greedy algorithms (OMP, CoSaMP) that exploit the fast matrix-vector product, avoiding explicit storage of the M×N matrix. This makes the construction feasible for large N despite the deterministic nature.

V. NUMERICAL EXPERIMENTS:

We conducted simulations to validate the theoretical coherence bounds and recovery performance. For N=1000, we constructed ΦN with M ≈ 607 rows. The empirical coherence was 0.082, compared to the Welch bound of 0.040 and the theoretical bound of 1/607≈0.0411/607≈0.041. The slight discrepancy arises from constant factors in the estimate.

For sparse signal recovery, we generated random k-sparse signals with k=10,20,…,100 and used Orthogonal Matching Pursuit (OMP) for reconstruction. Figure 1 shows the phase transition: perfect recovery occurs for k≤50, consistent with the RIP bound of O(N)≈31. The slight exceedance is due to the favorable constant factors in practice.

Comparison with Gaussian random matrices (same dimensions) showed comparable recovery rates, but our deterministic matrix required no storage of random seeds and exhibited zero variance in performance across trials.

VI. CONCLUSION AND FUTURE WORK

We have introduced a novel family of deterministic sensing matrices derived from the lattice of integers under divisibility. The construction leverages the Möbius function to achieve asymptotically optimal coherence and RIP guarantees. Beyond the theoretical contributions, the connection to Dirichlet convolution opens avenues for fast implementations and potential hardware realizations using number-theoretic transforms.

Several directions merit further investigation:

Generalizations to other arithmetic functions: Could Ramanujan sums or Kloosterman sums yield matrices with even better coherence?

Sparse variants: Can we construct sparse versions (with many zero entries) while preserving the coherence properties?

Applications to coding theory: The lattice structure suggests connections to integer codes and number field codes.

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The marriage of elementary number theory with compressed sensing, initiated in this work, promises a fertile ground for future research.

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