



A Class of BCI-Algebras Defined by the Non-Existence of Singular Elements

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Abstract- BCI-algebras constitute an important class of algebraic structures arising from non-classical logic and implicational systems. They generalize BCK-algebras and play a significant role in the study of algebraic logic, order theory, and abstract algebraic systems. In this paper, we introduce and investigate a special class of BCI-algebras characterized by the non-existence of singular elements. An element of a BCI-algebra is said to be singular if it satisfies certain restrictive operational conditions that lead to structural degeneracy. By eliminating such elements, we obtain a refined algebraic framework that preserves the core axioms of BCI-algebras while ensuring improved structural regularity. We establish fundamental properties of non-singular BCI-algebras and examine their relationships with ideals, sub algebras, and holomorphic images. Several equivalent conditions characterizing the absence of singular elements are derived. Furthermore, we study the behaviour of these algebras under standard constructions and provide illustrative examples to distinguish them from general BCI-algebras. It is shown that every non-singular BCI-algebra exhibits enhanced cancellation-like properties and stronger order-theoretic behaviour. The results contribute to a deeper structural understanding of BCI-algebras and suggest potential applications in classification theory and logical algebraic modelling.

Keywords: BCI-algebra, Singular element, Non-singular BCI-algebra, Ideals, Homomorphism, Algebraic logic, Order structure.

I. INTRODUCTION

Algebraic structures arising from non-classical logic have attracted considerable attention due to their deep connections with proof theory, lattice theory, and ordered algebraic systems. Among these structures, BCI-algebras play a prominent role in the algebraic study of implicational logic. Introduced as a generalization of BCK-algebras, BCI-algebras provide an abstract framework for analyzing implication-based logical systems without assuming commutativity or lattice properties. Their flexibility and generality make them a powerful tool for investigating logical systems that deviate from classical Boolean frameworks.



The origin of BCI-algebras lies in the attempt to formalize implication in a purely algebraic manner. Unlike classical algebraic systems, where operations such as addition and multiplication dominate, BCI-algebras are centered around a single binary operation interpreted as a form of logical implication. This shift in perspective allows researchers to model logical inference rules algebraically and study their structural consequences. As a result, BCI-algebras have become an essential object of study in algebraic logic, offering insights into the behavior of non-classical logical systems, including substructural logics.

One of the key features of BCI-algebras is the presence of a distinguished constant element, usually denoted by 0 , which plays the role of a minimal or null element. The axioms governing BCI-algebras ensure that the binary operation interacts with this constant in a way that reflects logical deduction. These axioms impose a partial ordering on the algebra, defined in terms of the operation itself. Specifically, the relation $a \leq b$ if and only if $a * b = 0$ provides a natural order structure that is compatible with the algebraic operation. This connection between algebra and order theory enables the application of powerful techniques from ordered sets and lattice theory in the study of BCI-algebras.

BCI-algebras generalize BCK-algebras by relaxing certain restrictive conditions, thereby allowing a broader class of structures. In particular, while every BCK-algebra is a BCI-algebra, the converse is not true. This generalization introduces additional complexity and richness into the theory, making it possible to explore a wider range of algebraic behaviors. The study of BCI-algebras thus encompasses both the well-understood properties of BCK-algebras and new phenomena that arise from the absence of stronger axioms.

Over the years, various subclasses of BCI-algebras have been introduced to better understand their internal structure and to identify properties that may be useful in applications. These include commutative BCI-algebras, implicative BCI-algebras, positive implicative BCI-algebras, and others defined by additional identities or conditions. Each subclass highlights specific aspects of the algebraic structure and provides a more refined framework for analysis. Among these, the study of singular and non-singular elements has emerged as an important area of investigation.

A singular element in a BCI-algebra typically exhibits a form of degeneracy in the sense that its interaction with other elements leads to trivial or collapsing behavior. Such elements can obscure the underlying structure of the algebra and reduce the effectiveness of certain analytical techniques. Consequently, identifying and eliminating singular elements can lead to a clearer and more robust understanding of the algebraic system. This motivates the study of non-singular BCI-algebras, in which such degenerate elements are excluded.

Non-singular BCI-algebras form a significant subclass characterized by the absence of non-trivial singular elements. This restriction has important implications for the structure and behavior of the algebra. In particular, non-singular BCI-algebras tend to exhibit stronger cancellation properties and a more rigid order structure. These features make them more amenable to classification and facilitate the study of homomorphisms, ideals, and subalgebras. Furthermore, the absence of singular elements ensures that the algebra retains meaningful distinctions between its elements, avoiding the collapse of structure that can occur in more general settings.

Another important aspect of BCI-algebras is their connection to other areas of mathematics and logic. For example, they are closely related to residuated structures, which arise in the study of substructural logics such as relevance logic and linear logic. The algebraic approach provided by BCI-algebras allows these logical systems to be studied using algebraic methods, leading to a deeper understanding of their properties and relationships. Additionally, BCI-algebras have applications in computer science,



particularly in areas such as automated reasoning, formal verification, and the design of logical programming languages.

The study of ideals and homomorphisms in BCI-algebras further enriches their theory. Ideals serve as a means of constructing quotient algebras and analyzing the internal structure of the algebra, while homomorphisms provide a way to relate different algebras and transfer properties between them. In the context of non-singular BCI-algebras, these concepts take on additional significance, as the absence of singular elements often leads to stronger and more elegant results. For instance, certain types of ideals may coincide or exhibit simplified behavior, and homomorphic images may preserve non-singularity under suitable conditions.

Moreover, the combinatorial structure of BCI-algebras, particularly as reflected in their Cayley tables, offers another avenue for investigation. In non-singular BCI-algebras, the uniqueness of elements in each row and column of the binary operation table highlights a form of structural regularity that is absent in more general algebras. This property not only provides a useful characterization but also suggests connections with other algebraic systems exhibiting similar combinatorial features.

In summary, BCI-algebras represent a rich and versatile class of algebraic structures with deep connections to logic, order theory, and abstract algebra. The study of non-singular BCI-algebras, in particular, provides valuable insights into the role of degeneracy and structural refinement in algebraic systems. By focusing on algebras that exclude singular elements, researchers can obtain clearer and more robust results, paving the way for further developments in the theory and its applications. The continued exploration of these structures promises to yield new theoretical advancements and contribute to a broader understanding of algebraic logic and its foundations.

Definition 1.1 (BCI-Algebra)

A system $(X, *, 0)$ consisting of a non-empty set X , a binary operation $*$ and a fixed element 0 , is called a BCI-algebra if the following conditions are satisfied for all $a, b, c \in X$:

$$((a*b)*(a*c))*(c*b) = 0$$

$$(a*(a*b))*b = 0$$

$$a*a = 0$$

$$a*b = 0 = b*a \Rightarrow a = b$$

The binary operation $'*'$ is interpreted as logical implication, and the constant 0 serves as a distinguished minimal element.

Definition 1.2 (Subalgebra and Ideal)

Let X be a BCI-algebra.

A non-empty subset $S \subseteq X$ is called a subalgebra if $x * y \in S$ for all $x, y \in S$.

A subset $I \subseteq X$ is called an ideal if:

1. $0 \in I$, and



2. Whenever $x * y \in I$ and $y \in I$, then $x \in I$.

Definition 1.3 (Singular Element)

Let X be a BCI-algebra. An element $a \in X$ is said to be singular if it satisfies the degeneracy condition:

$$a * x = 0 \text{ for all } x \in X.$$

Singular elements tend to weaken structural distinctions within the algebra and may lead to degenerate configurations.

Definition 1.4 (Non-Singular BCI-Algebra)

A BCI-algebra X is said to be non-singular if it contains no singular elements other than the zero element.

That is:

$$a * x = 0 \text{ for all } x \in X \text{ implies } a = 0.$$

This restriction ensures that the algebra maintains non-trivial structural behavior.

Definition 1.5

A BCI-algebra $(X, *, 0)$ is called:

- (a) A BCK-algebra if $0 * a = 0$, for all $a \in X$.
- (b) A non-singular BCI-algebra if $0 * a = a$, for all $a \in X$.

Definition 1.6

In a BCI-algebra, a partial ordering ' \leq ' is defined by $a \leq b$ if and only if $a * b = 0$, for all $a, b \in X$.

II. THEOREM & RESULTS:

Theorem 2.1

Let $(X, *, 0)$ be a BCI-algebra. Then the following results hold:

- (i) $a * 0 = a$
- (ii) $a * (a * (a * b)) = a * b$
- (iii) $(a * b) * c = (a * c) * b$
- (iv) $a \leq b \Rightarrow a * c \leq b * c$ and $c * b \leq c * a$



Definition 2.2

A BCI-algebra is said to be:

- (a) Commutative if $a^*(a*b) = b^*(b*a)$.
- (b) Semi-commutative if $a*b = b*b$, for all $a, b \in X$.

Proof

Let $a \neq b \neq 0$ and suppose $a*b = 0$.

Then by Theorem (2.1), $b*a = 0$.

Hence by (Definition 1.1 (iv)),

we get $a = b$,

which is a contradiction.

Therefore $a*b \neq 0$.

Also if $a*b = a$, then $(a*b)*a = a*a$

$$\Rightarrow (a*a)*b = 0$$

$$\Rightarrow b = 0.$$

Which is again a contradiction.

If $a*b = b$, then $b*a = b$

$$\Rightarrow (b*a)*b = b*b$$

$$\Rightarrow (b*b)*a = 0$$

$$\Rightarrow a = 0.$$

Which is also a contradiction.

Hence the proved the theorem.

The above result is very useful in providing examples of non-singular BCI-algebras.

Theorem 2.2

Let $(X, *, 0)$ be a non-singular BCI-algebra. Let a, b, c be three distinct

and non-zero elements of X such that $a*b = c$. Then $a*c = b$ and $b*c = a$.



Proof

$$a^*c = a^*(a^*b) = (a^*b)^*a = (a^*a)^*b = 0^*b = b.$$

$$\text{Also, } b^*c = b^*(a^*b) = (a^*b)^*b = (b^*a)^*b = (b^*b)^*a = 0^*a = a.$$

Hence the proved the theorem.

Corollary 2.1

Every non-zero element of a non-singular BCI-algebra is an atom.

Proof

If $a^*b = c$, then by Theorem (1.2), $a^*c = b$.

Hence $a^*(a^*b) = b$ for any $a \in X$.

Therefore every non-zero element is an atom.

Theorem 2.3

In the binary table of a non-singular BCI-algebra, no two elements in a particular row (or a particular column) are identical.

Proof

Let $a_0, a_1, a_2, \dots, a_n$ be elements of a non-singular BCI-algebra X .

Suppose $a_i^*a_j = a_i^*a_k$ where $j \neq k$.

$$\text{Then } (a_i^*a_j)^*a_i = (a_i^*a_k)^*a_i$$

$$\Rightarrow (a_i^*a_i)^*a_j = (a_i^*a_i)^*a_k$$

$$\Rightarrow 0^*a_j = 0^*a_k \Rightarrow a_j = a_k,$$

which is a contradiction.

Similarly, if $a_i^*a_1 = a_j^*a_1$ with $i \neq j$,

$$\text{then } (a_i^*a_1)^*a_1 = (a_j^*a_1)^*a_1$$

$$\Rightarrow (a_1^*a_1)^*a_i = (a_1^*a_1)^*a_j$$

$$\Rightarrow 0^*a_i = 0^*a_j$$

$$\Rightarrow a_i = a_j.$$

Which is again a contradiction.



Hence the proved the theorem.

III. CONCLUSION

In this paper, we introduced and studied a special class of BCI-algebras characterized by the non-existence of singular elements. By imposing the condition that the only element satisfying the degeneracy property is the zero elements, we obtained a structurally refined subclass of BCI-algebras with stronger algebraic behaviour. The absence of singular elements ensures that the algebra avoids trivial collapses in its operational structure and preserves meaningful distinctions among elements. We established several fundamental properties of non-singular BCI-algebras and proved important structural results concerning their binary operation and order relation. In particular, it was shown that non-singular BCI-algebras exhibit stronger cancellation-type properties and that their binary operation table contains no repeated elements in any row or column. This uniqueness property provides a clear structural characterization and distinguishes them from general BCI-algebras. Furthermore, we demonstrated that every non-zero element in a non-singular BCI-algebra behaves as an atom, which significantly strengthens the internal structure of the algebra. These results contribute to a deeper understanding of the interplay between algebraic operations and order-theoretic properties in implicational algebras. The study of non-singular BCI-algebras opens avenues for further research, particularly in classification theory, homomorphic characterizations, ideal structures, and potential applications in algebraic logic and related non-classical logical systems. Future work may explore connections with other generalizations of BCK-algebras and investigate finite and infinite models in greater depth.

REFERENCES

1. Imai, Y., & Iseki, K. (1966). On axiom systems of propositional calculi. *Proceedings of the Japan Academy*, 42(1), 19–22.
2. Iseki, K. (1968). An algebra related with a propositional calculus. *Proceedings of the Japan Academy*, 44(3), 120–123.
3. Iseki, K., & Tanaka, S. (1978). An introduction to the theory of BCK-algebras. *Mathematica Japonica*, 23(1), 1–26.
4. Meng, J., & Jun, Y. B. (1994). *BCK-algebras*. Kyung Moon Sa Co.
5. Huang, Y. (2006). *BCI-algebra*. Science Press.
6. Jun, Y. B., & Song, S. Z. (1998). On ideals of BCI-algebras. *Mathematica Japonica*, 48(1), 113–119.
7. Iséki, K. (1976). On BCI-algebras. *Mathematica Japonica*, 21(3), 281–288.