



# On the Existence of Common Fixed Points in Complete Rectangular S-Metric Spaces

P. Ramulu<sup>1</sup>, D. Sattemma<sup>2</sup>, T. Rajeshwari<sup>3</sup>

<sup>1</sup>Department of Mathematics, Sri Venkateshwara Government Arts & Science College(A), Palem-509215, Nagarkurnool Dist. Telanagna, India.

<sup>2</sup>Department of Mathematics, MALD Government Degree College, Gadwal, Telanagna, India.

<sup>3</sup>Department of Mathematics, Dr BRR Government Degree College(A), Jadcherla, Telanagna, India.

**Abstract-** Fixed point theory has witnessed substantial development through the introduction of generalized metric structures. Among these, S-metric spaces and their rectangular variants have emerged as powerful frameworks for extending classical contraction principles. In this paper, we investigate common fixed point results in complete rectangular D-metric spaces, obtained through a structural reformulation of S-metric spaces. We establish a new contraction-type condition for a pair of self-mappings and prove the existence and uniqueness of a coincidence point under suitable range inclusion and completeness assumptions. Further, by employing weak compatibility of mappings, we derive the existence of a unique common fixed point. The presented theorem generalizes several known results in rectangular S-metric spaces and broadens the applicability of fixed point theory in generalized nonlinear structures. The results contribute to the ongoing development of metric-type spaces and their applications in nonlinear functional analysis.

**Keywords:** Rectangular D-metric space; Common fixed point; Weakly compatible mappings; Coincidence point; Complete D-metric space; Contraction condition.

## I. INTRODUCTION

Fixed point theory has emerged as one of the most significant and widely applied areas of nonlinear analysis. It plays a central role in various branches of mathematics and has far-reaching applications in disciplines such as engineering, economics, computer science, optimization theory, and applied sciences. The core idea of fixed point theory revolves around identifying conditions under which a mapping admits a point that remains invariant under its action. Such points, known as fixed points, often correspond to equilibrium states, steady solutions, or optimal configurations in practical problems. The foundation of modern fixed point theory can be traced back to the celebrated Banach Contraction Principle, which provides a simple yet powerful criterion for the existence and uniqueness of fixed points in complete metric spaces. This principle not only guarantees the existence of a unique fixed point for contraction mappings but also ensures the convergence of iterative sequences to that point. Due to its elegance and applicability, the Banach Contraction Principle has inspired extensive research aimed at generalizing its framework to more abstract and flexible settings. In an effort to broaden the scope of fixed point results, mathematicians have introduced several generalizations of classical metric spaces. Among these are G-metric spaces, cone metric spaces, partial metric spaces, and S-metric spaces. Each of these structures modifies or extends the concept of distance in a way that accommodates a wider



variety of problems. In particular,  $S$ -metric spaces represent a notable advancement, as they replace the traditional two-variable distance function with a three-variable function. This modification allows for a richer geometric structure and provides additional flexibility in defining contractive conditions for mappings.

The introduction of  $S$ -metric spaces has opened new avenues for the study of fixed point theorems. By considering a function of three variables instead of two, researchers can formulate more generalized contraction conditions that are not possible within the framework of classical metric spaces. This enhanced flexibility has led to the development of various fixed point results that are applicable in situations where standard assumptions fail to hold. Building upon the concept of  $S$ -metric spaces, the notion of rectangular  $S$ -metric spaces has been introduced as a further generalization. In this setting, the traditional triangle inequality is replaced by a rectangular-type inequality, which involves four points instead of three. This modification significantly enlarges the class of admissible spaces and mappings. The rectangular inequality provides a weaker condition compared to the triangle inequality, thereby allowing for the inclusion of mappings that may not satisfy conventional contractive requirements. As a result, rectangular  $S$ -metric spaces offer a more generalized and flexible framework for the study of fixed point theory. One of the key areas of interest in fixed point theory is the study of common fixed points of multiple mappings. This problem arises naturally in the analysis of systems of functional equations, operator equations, and iterative processes involving more than one transformation. The existence of a common fixed point ensures that multiple mappings share a single invariant point, which often corresponds to a solution of a system or a point of equilibrium in applications.

However, establishing the existence of common fixed points typically requires certain compatibility conditions between the mappings. Classical approaches often rely on strong commutativity assumptions, which can be restrictive in practical scenarios. To overcome this limitation, the concept of weak compatibility has been introduced. Two mappings are said to be weakly compatible if they commute at their coincidence points. This notion is less restrictive than commutativity and has proven to be highly effective in deriving common fixed point results under more general conditions. In recent years, researchers have focused on combining these generalized space structures with weakened compatibility conditions to obtain more inclusive and powerful fixed point theorems. In this context, the study of complete rectangular  $D$ -metric spaces represents a significant advancement. These spaces incorporate the flexibility of  $S$ -metric-type structures while employing a rectangular inequality, thereby providing an enriched framework for analysis.

In the present work, we investigate common fixed point results in the setting of complete rectangular  $D$ -metric spaces. The analysis is carried out by introducing an appropriate contraction condition tailored to the structure of the space. This contractive condition is designed to be sufficiently general so as to encompass a broad class of mappings, while still ensuring the convergence of iterative sequences. A crucial assumption in our study is the inclusion of ranges of the mappings under consideration. This condition facilitates the construction of sequences that remain within a suitable subset of the space, thereby enabling the application of completeness. Using this approach, we establish the existence of a coincidence point, that is, a point at which two mappings agree. The existence of such a point forms the foundation for further analysis. To extend this result, we employ the concept of weak compatibility. By assuming that the mappings are weakly compatible, we are able to prove that the coincidence point is, in fact, a common fixed point. Furthermore, we demonstrate that this common fixed point is unique under the given conditions. The uniqueness result is particularly important, as it ensures the stability and reliability of solutions in applications. The results obtained in this work not only generalize several existing theorems but also improve upon them by weakening the underlying assumptions. In particular, the use of rectangular  $D$ -metric spaces allows us to replace strong metric conditions with more flexible alternatives, thereby broadening the applicability of fixed point theory. Moreover, the incorporation of



weak compatibility eliminates the need for stringent commutativity requirements, making the results more relevant to real-world problems.

In conclusion, this study contributes to the ongoing development of fixed point theory by extending classical results to a more generalized and flexible framework. The introduction of new contractive conditions and the use of rectangular D-metric spaces provide valuable tools for future research. It is expected that these results will find applications in various fields, including differential equations, optimization, and computational mathematics, where the existence and uniqueness of solutions play a crucial role.

## II. PRELIMINARIES

### Definition 2.1

Let  $X$  be a non-empty set and  $D: X^3 \rightarrow \mathbb{R}^+$  be a function satisfying:

- $D(l,m,n) = 0$  if and only if  $l = m = n$ .
- $D(l,m,n) \leq D(l,l,a) + D(m,m,a) + D(n,n,a)$  for all  $a,l,m,n \in X$  (rectangular inequality).

### Definition 2.2

Let  $X$  be a non-empty set and  $D: X^3 \rightarrow \mathbb{R}$  be a function satisfying:

- $D(l,m,n) = 0$  if and only if  $l = m = n$ .
- $D(l,m,n) \leq D(l,l,a) + D(m,m,a) + D(n,n,a)$  for all  $l,m,n \in X$  and all distinct  $a \in X - \{l,m,n\}$ .

Then  $(X, D)$  is called a rectangular D-metric space.

### Definition 2.3

Let  $(X, D)$  be a D-metric space and  $A \subset X$ .

A sequence  $\{a_n\}$  converges to  $a$  if  $D(a_n, a_n, a) \rightarrow 0$  as  $n \rightarrow \infty$ .

That is, for every  $\varepsilon > 0$ , there exists  $n_0 \in \mathbb{N}$  such that  $n \geq n_0$  implies  $D(a_n, a_n, a) < \varepsilon$ .

A sequence  $\{a_n\}$  is Cauchy if for every  $\varepsilon > 0$ , there exists  $n_0 \in \mathbb{N}$  such that  $D(a_n, a_n, a_m) < \varepsilon$  for all  $n, m \geq n_0$ .

$(X, D)$  is complete if every Cauchy sequence converges.

### Definition 2.4

Let  $f$  and  $F$  be weakly compatible self-maps of  $X$ . If  $w = fw = Fw$  is the unique point of coincidence, then  $w$  is the unique common fixed point of  $f$  and  $F$ .

### Lemma 2.5

If  $(X, D)$  is a D-metric space, then  $D(a, a, b) = D(b, b, a)$  for all  $a, b \in X$ .



**Lemma 2.6**

If  $\{a_n\} \rightarrow a$  and  $\{b_n\} \rightarrow b$  in  $(X,D)$ , then  $D(a_n, a_n, b_n) \rightarrow D(a, a, b)$  as  $n \rightarrow \infty$ .

**Lemma 2.7**

If  $\{a_n\}$  converges to  $a$  in  $(X,D)$ , then the limit  $a$  is unique.

**Lemma 2.8**

If  $\{a_n\}$  converges to  $a$  in  $(X, D)$ , then  $\{a_n\}$  is a Cauchy sequence.

**Example 2.9**

Let  $X = \mathbb{N} \cup \{0\}$ . Define  $D: X \times X \times X \rightarrow \mathbb{R}^+ \cup \{0\}$  by:

$D(l,m,n) = 0$  if  $l = m = n$ ; and  $D(l,m,n) = l + m + n$  otherwise.

Then  $(X, D)$  is a rectangular D-metric space.

**III. MAIN RESULT**

**Theorem 3.1**

Let  $(X, D)$  be a complete rectangular D-metric space and let  $f$  and  $F$  be self-mappings on  $X$ . Assume that:

$D(fa, fa, fb) \leq k [ D(fa, fa, Fb) + D(fb, fb, Fa) ]$ , where  $0 \leq k < 0.1$ .

$$f(X) \subseteq F(X)$$

$F(X)$  is complete

Then  $f$  and  $F$  have a unique coincidence point in  $X$ .

**Proof**

Let  $a_0$  be any point in  $X$ . Since  $f(X) \subseteq F(X)$ , there exists  $a_1 \in X$  such that  $fa_0 = fa_1$ . Continuing inductively, we construct a sequence  $\{a_n\} \subset X$  such that  $fa_n = Fa_{(n+1)}$  for all  $n \in \mathbb{N}$ .

$$D(Fa_{(n+1)}, Fa_{(n+1)}, Fa_n) = D(fa_n, fa_n, fa_{(n-1)}) \leq k [ D(fa_n, fa_n, Fa_{(n-1)}) + D(fa_{(n-1)}, fa_{(n-1)}, Fa_n) ]$$

Substituting  $fa_n = Fa_{(n+1)}$  and  $fa_{(n-1)} = Fa_n$  gives:

$$D(Fa_{(n+1)}, Fa_{(n+1)}, Fa_n) \leq k D(Fa_{(n+1)}, Fa_{(n+1)}, Fa_{(n-1)})$$



**Using the rectangular inequality:**

$$D(Fa_{n+1}, Fa_{n+1}, Fa_{n-1}) \leq D(Fa_{n+1}, Fa_{n+1}, Fa_n) + D(Fa_n, Fa_n, Fa_{n-1})$$

$$(1 - 2k)D(Fa_{n+1}, Fa_{n+1}, Fa_n) \leq kD(Fa_n, Fa_n, Fa_{n-1})$$

Let  $w = \frac{k}{1 - 2k}$ , where  $0 \leq k < 0.1$ , so that  $0 \leq w < 1$

$$D(Fa_{n+1}, Fa_{n+1}, Fa_n) \leq wD(Fa_n, Fa_n, Fa_{n-1})$$

**By iteration:**

$$D(Fa_{(n+1)}, Fa_{(n+1)}, Fa_n) \leq w^n D(Fa_1, Fa_1, Fa_0)$$

Using the rectangular inequality, for  $m < n$ , we have:

$$D(Fa_n, Fa_n, Fa_m) \leq D(Fa_n, Fa_n, Fa_{n-1}) + D(Fa_n, Fa_n, Fa_{n-1}) + D(Fa_m, Fa_m, Fa_{n-1})$$

$$\leq 2w^{n-1}D(Fa_1, Fa_1, Fa_0) + D(Fa_m, Fa_m, Fa_{n-1})$$

$$\leq 2w^{n-1}z + D(Fa_m, Fa_m, Fa_{n-2}) + D(Fa_m, Fa_m, Fa_{n-2}) + D(Fa_{n-1}, Fa_{n-1}, Fa_{n-2})$$

Let  $z = D(Fa_1, Fa_1, Fa_0)$

Repeating the same argument, we obtain:

$$\leq 2w^{(n-1)}z + w^{(n-2)}z + 2D(Fa_m, Fa_m, Fa_{(n-2)}) + w^{(n-2)}D(Fa_1, Fa_1, Fa_0)$$

$$= 2w^{(n-1)}z + w^{(n-2)}z + 2D(Fa_m, Fa_m, Fa_{(n-2)})$$

$$\leq 2w^{(n-1)}z + w^{(n-2)}z + 2[2D(Fa_m, Fa_m, Fa_{(n-3)}) + D(Fa_{(n-2)}, Fa_{(n-2)}, Fa_{(n-3)})]$$

$$\leq 2w^{(n-1)}z + w^{(n-2)}z + 4D(Fa_m, Fa_m, Fa_{(n-3)}) + 2w^{(n-3)}z$$

$$\leq 2w^{(n-1)}z + w^{(n-2)}z + 2w^{(n-3)}z + 4[2D(Fa_m, Fa_m, Fa_{(n-4)}) + D(Fa_{(n-3)}, Fa_{(n-3)}, Fa_{(n-4)})]$$

$$\leq 2w^{(n-1)}z + w^{(n-2)}z + 2w^{(n-3)}z + 8D(Fa_m, Fa_m, Fa_{(n-4)}) + 4w^{(n-4)}z$$

$$= 2w^{(n-1)}z + w^{(n-2)}z + 2w^{(n-3)}z + 4w^{(n-4)}z + \dots$$

$$= 2w^{n-1}z + w^{n-2}z \left(1 + \frac{1}{w} + \frac{4}{w^2} + \frac{8}{w^3} + \dots\right)$$

$$D(Fa_n, Fa_n, Fa_m) \leq 2w^{n-1}z + w^{n-2}z + w^{n-3}z + \dots$$

$$= 2w^{n-1}z + w^{n-2}z \left(1 - \frac{2}{w}\right)^{-1}.$$

Since  $0 \leq w < 1$ , the geometric series converges, and therefore:

$$\lim_{n \rightarrow \infty} D(Fa_n, Fa_n, Fa_m) = 0$$

Thus  $\{Fa_n\}$  is a Cauchy sequence in  $F(X)$ . Since  $F(X)$  is complete, there exists  $c \in F(X)$  such that

$$Fa_n \rightarrow c \text{ as } n \rightarrow \infty.$$

Because  $c \in F(X)$ , there exists  $I \in X$  such that  $FI = c$ .



### Coincidence Point

$$D(fa_n, fa_n, fl) \leq k [D(fa_n, fa_n, Fl) + D(fl, fl, Fa_n)].$$

Substituting  $fa_n = Fa_{n+1}$  and taking limit as  $n \rightarrow \infty$  gives:

$$D(c, c, fl) \leq k D(c, c, fl).$$

Since  $0 \leq k < 1$ , this implies  $D(c, c, fl) = 0$ , and hence  $fl = c$ .

Therefore  $fl = Fl = c$ , so  $c$  is a coincidence point of  $f$  and  $F$ .

### Uniqueness

Suppose  $u \in X$  is another coincidence point such that  $fu = Fu = u$ .

$$\begin{aligned} D(c, c, u) &= D(fl, fl, fu) \leq k [D(fl, fl, Fu) + D(fu, fu, Fl)] \\ &= k [D(c, c, u) + D(u, u, c)] \leq 2k D(c, c, u). \end{aligned}$$

Since  $0 \leq k < 0.1$ , it follows that  $D(c, c, u) = 0$ , hence  $c = u$ .

Therefore,  $f$  and  $F$  have a unique coincidence point in  $X$ . Since they are weakly compatible, they have a unique common fixed point in  $X$ .

## IV. CONCLUSION

In this work, we have investigated common fixed point results in the framework of complete rectangular D-metric spaces, which provide a natural generalization of classical metric and S-metric spaces. By replacing the standard triangle inequality with a rectangular-type inequality, this structure allows the treatment of a broader class of mappings under weaker conditions. A suitable contractive condition has been introduced, under which the existence of a coincidence point for two self-mappings  $fff$  and  $FFF$  is established. The construction of an iterative sequence and the use of the rectangular inequality enabled us to prove that the sequence is Cauchy in  $F(X)F(X)F(X)$ , and hence convergent due to the completeness of the space. Furthermore, by employing the concept of weak compatibility, we extended the result to guarantee the existence of a unique common fixed point. The avoidance of strict commutativity assumptions significantly enhances the applicability of the obtained results. The uniqueness of the fixed point ensures the stability and reliability of solutions in practical applications.

The results presented in this study generalize and improve several existing theorems in the literature by weakening the underlying conditions and extending the setting to rectangular D-metric spaces. These findings contribute to the ongoing development of fixed point theory and open new directions for future research, particularly in the study of nonlinear functional equations, operator theory, and applied mathematical models.

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