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# Properties of an Unbounded Solutions of a Fourth Order Neutral Dynamic Equations

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**Abstract- In this paper, the author studied, the properties of an unbounded solutions of a class of fourth order dynamic equation of the form**

$$\left(r(t)\left(x(t)+p(t)x(\alpha(t))\right)^{\Delta^2}\right)^{\Delta^2}+g(t)G\left(x(\beta(t))\right)-h(t)H\left(x(\gamma(t))\right)=0 \qquad (H)$$

**for $t\in[t_0,\infty)_{\mathbb{T}}$ where $\mathbb{T}$ is a time scale with $\sup\mathbb{T}=+\infty$, $t_0(\geq 0)\in\mathbb{T}$ are studied under the assumption**

$H_0$ $\int_{t_0}^{\infty}\frac{t}{r(t)}\Delta t=\infty$ **and** $((H_0{}'$ $\int_{t_0}^{\infty}\frac{t}{r(t)}\Delta t<\infty$ **for various ranges of** $p(t)$.

**Keywords- dynamic equation, delay, oscillation, positive and negative coefficients, bounded solution.**

## I. INTRODUCTION

In [3] and [4], Parhi and Tripathy have discussed the oscillatory and asymptotic properties of solutions of fourth order differential equation of the form

$$(r(t)(y(t)+p(t)y(t-\alpha))'')''+q(t)G\big(y(t-\sigma)\big)=0,\ \ t\in[0,\infty)$$

where $r,q\in C([0,\infty),(0,\infty))$, $G\in C(R,R)$ , $p\in C((0,\infty),R)$ and $\alpha,\sigma$ are positive real constants; are studied under the assumption

$$\int_0^{\infty}\frac{t}{r(t)}dt=\infty \ \text{ or } \ \int_0^{\infty}\frac{t}{r(t)}dt<\infty.$$

In [5] and [6], Panigrahi, Graef and Ramireddy have discussed the oscillatory and asymptotic properties of solutions of fourth order dynamic equation of the form

$$(r(t)(y(t)+p(t)y(\alpha(t)))^{\Delta^2})^{\Delta^2}+q(t)G\left(x(\beta(t))\right)-h(t)H\left(x(\gamma(t))\right)=0$$

and

$(r(t)(y(t)+p(t)y(\alpha(t)))^{\Delta^2})^{\Delta^2}+q(t)G\left(x(\beta(t))\right)-h(t)H\left(x(\gamma(t))\right)=f(t)$, $t\in[t_0,\infty)_T$, where $T$ is a timescale,


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$r, q \in C_{rd}([t_0,\infty),(0,\infty))$, $p, h, f \in C_{rd}([t_0,\infty), R)$, $G, H \in C(R, R)$ and $\alpha, \beta, \gamma \in C_{rd}(T, T)$ are increasing functions such that

$\alpha(t) \leq t, \beta(t) \leq t, \gamma(t) \leq t$ with $\lim_{t\to\infty} \alpha(t) = \lim_{t\to\infty} \beta(t) = \lim_{t\to\infty} \gamma(t) = \infty$; are studied under the assumption

$\int_{t_0}^{\infty} \frac{t}{r(t)} \Delta t = \infty$ or $\int_{t_0}^{\infty} \frac{\sigma(t)}{r(t)} \Delta t < \infty$.

When we are concerned with oscillation of all solutions of the following equation

$$(p(t)(u(t) + r(t)u(\alpha(t)))^{\Delta^2})^{\Delta^2} + g(t)G\left(u(\beta(t))\right) = 0$$

in [7] and [8] by Tripathy under two sorts of assumptions

$\int_{t_0}^{\infty} \frac{t}{p(t)} \Delta t = \infty$ or $\int_{t_0}^{\infty} \frac{t}{p(t)} \Delta t < \infty$.

Similarly, in an another work [9 ] of Grace and Zafer, we can find all solutions of oscillatory problem for the dynamic equations

$$(p(t)((u(t) + r(t)u(\alpha(t)))^{\Delta^3})^{\mu_1})^{\Delta} + g(t)u^{\mu_2}(\beta(t)) = 0$$

under the following restrictions:

(i) $\mu_1$ and $\mu_2$ are the quotient of odd positive integers such that $\mu_2 \leq \mu_1$,

$(ii)$ $|r(t)| \leq 1$

and (iii) $\int_{t_0}^{\infty} p^{-\frac{1}{\mu_1}}(t)\Delta t = \infty$, $t_0 > 0$.

A time scale is an arbitrary nonempty closed subset of real number. We denote a timescale by $T$, the forward jump operator $\sigma$ is defined as $\sigma(t) = \inf\{s \in T: s > t\}$, the delta derivative of $f$ is denoted by $f^{\Delta}$, the useful double integration formulas are

$$\int_v^u \int_v^t f(\theta)\Delta\theta \, \Delta s = \int_v^u (u - \sigma(t)) f(t) \Delta t$$

and

$\int_v^u \int_t^u f(\theta)\Delta\theta \, \Delta s = \int_v^u (\sigma(t) - v) f(t)\Delta t.$

For more basics in time scale refer the books [1, 2]. In this work, we assume the following assumptions on $G$, $\alpha$ and $\beta$.

$(H_1)$ $G(xy) \geq G(x)G(y)$ for $x, y \in R$ and $x, y > 0$;

$(H_2)$ $G(-x) = -G(x)$ for $\in R$ ;

$(H_3)$ there exists $\lambda > 0$ such that $G(x) + G(y) \geq \lambda G(x + y)$ for $x, y \in R$ and $x, y > 0$;

$(H_4)$ $\alpha(\beta(t)) = \beta(\alpha(t))$ forall $t \in [t_0, \infty)_T$.

**Remark 1.** The prototype of $G$ satisfying $(H_1)$, $(H_2)$ and $(H_3)$ could be of the form

$G(u) = (a^* + b^*|u|^{\gamma})|u| sgn\, u$ And $G(u) = |u|^{\eta} sgn\, u$

where $a^* \geq 1, b^* \geq 1, \gamma \geq 0$ and $\eta > 0$.

Let $t_{-1} = \inf_{t\in[t_0,\infty)_{\mathbb{T}}} \{\alpha(t), \beta(t), \gamma(t)\}$. By a solution of $(H)$, we mean a function $x \in C_{rd}([t_{-1},\infty)_{\mathbb{T}}, \mathbb{R})$ such that $x(t) + p(t)x(\alpha(t)) \in C_{rd}{}^2([t_0,\infty)_T, R)$, $r(t)(x(t) + p(t)x(\alpha(t)))^{\Delta^2} \in C_{rd}{}^2([t_0,\infty)_T, R)$ and satisfies $(H)$ identically on $[t_0, \infty)_T$.

. A solution of $(H)$ is said to be oscillatory if there exists a sequence $\{s_n\}$ in $[t_0,\infty)_{\mathbb{T}}$ such that $(s_n)x(\sigma(s_n)) \leq 0$ ; Otherwise, it is called non oscillatory.

## II. SOME PREPARATORY RESULTS

In this section, we present some preparatory results which are useful for our discussion. Throughout our text, we use the quasi-derivative operators on $D$ as follows.

$$D_0 u = u,\ D_1 u = D_0 u^{\Delta},\ D_2 u = r D_1{}^{\Delta} u,\ D_3 u = D_2{}^{\Delta} u,\ \text{and } D_4 u = D_3{}^{\Delta} u.$$

**Lemma 2.1**. Let $(H_0)$ hold and $u$ be a real valued function on $[t_0,\infty)_{\mathbb{T}}$ such that $D_4 u(t) \leq 0$ for large $t$. If $u(t) > 0$ ultimately, then one of the Cases $(a)$ and $(b)$ holds for large $t$ and if $u(t) < 0$ ultimately, then one of the Cases $(b), (c), (d)$ and $(e)$ holds for large $t$, where

$(a)$ $D_1u(t) > 0$, $D_2u(t) > 0$ and $D_3u(t) > 0$,
$(b)$ $D_1u(t) > 0$, $D_2u(t) < 0$ and $D_3u(t) > 0$,
$(c)$ $D_1u(t) < 0$, $D_2u(t) < 0$ and $D_3u(t) > 0$,
$(d)$ $D_1u(t) < 0$, $D_2u(t) < 0$ and $D_3u(t) < 0$,
$(e)$ $D_1u(t) < 0$, $D_2u(t) > 0$ and $D_3u(t) > 0$,

**Lemma 2.2**. Let $(H_0{}')$ hold and $u$ be a real valued function on $[t_0,\infty)_{\mathbb{T}}$ such that $D_4u(t) \le 0$ for large $t$. If $u(t) > 0$ ultimately, then one of the Cases $(a)$ - $(d)$ holds for large $t$ and if $u(t) < 0$ ultimately, then one of the Cases $(b) - (f)$ holds for large $t$, where
$(a)$ $D_1u(t) > 0$, $D_2u(t) > 0$ and $D_3u(t) > 0$,
$(b)$ $D_1u(t) > 0$, $D_2u(t) < 0$ and $D_3u(t) > 0$,
$(c)$ $D_1u(t) > 0$, $D_2u(t) < 0$ and $D_3u(t) < 0$,
$(d)$ $D_1u(t) < 0$, $D_2u(t) > 0$ and $D_3u(t) > 0$,
$(e)$ $D_1u(t) < 0$, $D_2u(t) < 0$ and $D_3u(t) > 0$,
$(f)$ $D_1u(t) < 0$, $D_2u(t) < 0$ and $D_3u(t) < 0$.

## III. UNBOUNDED OSCILLATION RESULTS

In this section, unbounded oscillation criteria for $(H)$ are established under the assumptions $(H_0)$ and $(H_0{}')$. Before stating our main results, we assume the following hypothesis for our use in the sequel :
$B[u,v] = \int_v^u \frac{(u-\sigma(t))(t-v)}{r(t)}\Delta t$, $u > t > v$;
$C[u,v] = \int_v^u \frac{(\sigma(t)-v)(u-t)}{r(t)}\Delta t$, $u > t > v$;
and $D[u,v] = \int_v^u \frac{(\sigma(t)-v)(t-v)}{r(t)}\Delta t$, $u > t > v$;

**Theorem 3.1.** Let $0 \le p(t) \le p < \infty$ and $\beta(t) \le \alpha(\alpha(t))$, $\gamma^{-1}(t) \le \beta(t)$. Suppose that, $(H_0{}')$ and $(H_1) - (H_4)$ hold. If
$(H_5)$ $\int_T^\infty \frac{\sigma(s)}{r(s)} \int_s^\infty \sigma(t)h(t)\Delta t\Delta s < \infty$;
$(H_6)$ $\frac{G(u)}{u} \ge M_1 > 0$ for $u \ne 0$;
$(H_7)$ $\limsup_{s\to\infty} \int_s^{\beta^{-1}(\alpha(s))} Q(t)G\left[B\big(\beta(t),\beta(s)\big)\right]\Delta t > \frac{1+G(a)}{\lambda M_1}$;
$(H_8)$ $\limsup_{s\to\infty} \int_s^{\beta^{-1}(\alpha(s))} Q(t)G\left[C\big(\beta(t),\beta(s)\big)\right]\Delta t > \frac{1+G(a)}{\lambda M_1}$;
$(H_9)$ $\limsup_{s\to\infty} \int_s^{\gamma^{-1}(s)} Q(t)G\left[D\big(\gamma^{-1}(t),\gamma^{-1}(s)\big)\right]\Delta t > \frac{1+G(a)}{\lambda M_1}$;
$(H_{10})$ $\limsup_{s\to\infty} \int_{\beta(s)}^{s} Q(t)G\left[D\big(\beta(s),\beta(t)\big)\right]\Delta t > \frac{1+G(a)}{\lambda M_1}$;
hold, where $Q(t) = \min\{g(t), g(\alpha(t))\}$ for $t \ge t_{-1}$, then every unbounded solution of $(H)$ oscillates.

**Proof:** Assume the contrary that, $x(t)$ be an unbounded non-oscillatory solution of (H) such that $x(t) > 0$ for $t \ge t_0$. Then there exists $t_1 \in [t_0,\infty)_{\mathbb{T}}$ such that $x(t)$, $x\big(\alpha(t)\big)$, $x\big(\beta(t)\big)$ *and* $x(\gamma(t))$ are all positive for $t \ge t_1$. Setting

$$y(t) = x(t) + p(t)x\big(\alpha(t)\big) \tag{3.1}$$

$$k(t) = \int_t^\infty \frac{(\sigma(s)-t)}{r(s)} \int_s^\infty (\sigma(\theta)-s)h(\theta)H(x(\gamma(\theta)))\Delta\theta\Delta s \tag{3.2}$$

$$z(t) = y(t) - k(t) = x(t) + p(t)x\big(\alpha(t)\big) - k(t) \tag{3.3}$$

$$D_4z(t) = -g(t)G(x(\beta(t))) \le 0 \tag{3.4}$$

forall $t \ge t_1$. Then there exists $t_2 \in [t_1,\infty)_T$ such that $D_iz(t), i = 0,1,2,3$ are eventually of one sign on $[t_2,\infty)_T$. Assume that, $z(t) > 0$. In what follows, we consider the Cases $(a) - (d)$ of Lemma 2.2. Let Case $(a)$ holds. For $u > s > v > t_2$,

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$$D_2z(u) - D_2z(v) = \int_v^u D_3z(s)\Delta s \geq (u-v)D_3z(u),$$

implies

$$D_2z(u) \geq (u-v)D_3z(u)$$

or

$$r(u){D_1}^{\Delta}z(u) \geq (u-v)D_3z(u)$$

Or

$${D_1}^{\Delta}z(u) \geq \frac{(u-v)}{r(u)}D_3z(u)$$

For $u > \theta > v > t_2$, we have

$$D_1z(u) - D_1z(v) = \int_v^u \frac{(\theta - v)}{r(\theta)}D_3z(\theta)\Delta s,$$

Implies

$$D_1z(u) \geq D_3z(u)\int_v^u \frac{(\theta - v)}{r(\theta)}\Delta\theta,$$

For $u > t > v > t_2$, we have

$$z(u) - z(v) \geq \int_v^u D_3z(t)\int_v^t \frac{(\theta - v)}{r(\theta)}\Delta\theta\, \Delta t,$$

Or

$$z(u) \geq D_3z(u)\int_v^u\int_v^s \frac{(\theta - v)}{r(\theta)}\Delta\theta\, \Delta s$$

Or

$$z(u) \geq D_3z(u)\int_v^u \frac{(u-\sigma(t))(t-v)}{r(t)}\Delta s$$

Or

$$z(u) \geq D_3z(u)B[u,v] \tag{3.5}$$

From (3.4) and for $\beta^{-1}(\alpha(s)) > t > s > t_2$ , we have

$0 = D_4z(t) + g(t)G\left(x(\beta(t))\right) + G(p)\left[D_4z(\alpha(t)) + g(\alpha(t))G\left(x\left(\beta(\alpha(t))\right)\right)\right]$

$\geq D_4z(t) + G(p)D_4z(\alpha(t)) + Q(t)\ [G\left(x(\beta(t))\right) + G(p)G\left(x(\beta(\alpha(t)))\right)]$

$\geq D_4z(t) + G(p)D_4z(\alpha(t)) + Q(t)[G\left(x(\beta(t))\right) + G(p(t))G(x(\alpha(\beta(t)))]$

$\geq D_4z(t) + G(p)D_4z(\alpha(t)) + Q(t)[G\left(x(\beta(t))\right) + G(p(t))G(x(\alpha(\beta(t))))]$

$\geq D_4z(t) + G(p)D_4z(\alpha(t)) + Q(t)\lambda[G\left(x(\beta(t)) + p(t)x(\alpha(\beta(t)))\right)]$

$\geq D_4z(t) + G(p)D_4z(\alpha(t)) + \lambda Q(t)G(y(\beta(t)))$

$\geq D_4z(t) + G(p)D_4z(\alpha(t)) + \lambda Q(t)G(z(\beta(t)))$

$\geq D_4z(t) + G(p)D_4z(\alpha(t)) + \lambda Q(t)G[D_3z(\beta(t))]G(B[\beta(t),\beta(s)]))$ (3.6)

Integrating the above inequality, from $s$ to $\beta^{-1}(\alpha(s))$, we obtain

$$\lambda \int_s^{\beta^{-1}(\alpha(s))} Q(t)G(D_3z(\beta(t))B[\beta(t),\beta(s)])\Delta t \leq D_3z(s) + G(p)D_3z(\alpha(s))$$

implies

$$\lambda G(D_3z(\alpha(s))\int_s^{\beta^{-1}(\alpha(s))} Q(t)G[B[\beta(t),\beta(s)]]\Delta t \leq D_3z(\alpha(s)) + G(p)D_3z(\alpha(s)) \quad \leq (1+G(p))D_3z(\alpha(s))$$

Or

$\int_s^{\beta^{-1}(\alpha(s))} Q(t)G(B[\beta(t),\beta(s)])\Delta t \leq \frac{(1+G(p))D_3z(\alpha(s))}{\lambda G[D_3z(\alpha(s))]} \leq \frac{(1+G(p))}{\lambda M_1}$,

where $\beta(s) \leq \alpha(s)$.

Case $(b)$. For $u > s > v > t_2$,

$$D_2z(u) - D_2z(v) = \int_v^u D_3z(s)\Delta s \geq (u-v)D_3z(u),$$

implies

$$-D_2z(v) \geq (u-v)D_3z(u)$$

Or

$$-r(v){D_1}^{\Delta}z(v) \geq (u-v)D_3z(u)$$

Or

$$-{D_1}^{\Delta}z(v) \geq \frac{(u-v)}{r(v)}D_3z(u)$$

For $u > \theta > v > t_2$, we have

$$-D_1z(u) + D_1z(v) = \int_v^u \frac{(u-\theta)}{r(\theta)}D_3z(u)\Delta s,$$

implies

$$D_1z(v) \geq D_3z(u)\int_v^u \frac{(u-\theta)}{r(\theta)}\Delta\theta,$$

For $u > t > v > t_2$, we have

$$z(u) - z(v) \geq \int_v^u D_3z(u)\int_t^u \frac{(u-\theta)}{r(\theta)}\Delta\theta\ \Delta t,$$

Or

$$z(u) \geq D_3z(u)\int_v^u\int_t^u \frac{(u-\theta)}{r(\theta)}\Delta\theta\ \Delta t$$

Or

$$z(u) \geq D_3z(u)\int_v^u \frac{(\sigma(t)-v)(u-t)}{r(t)}\Delta t$$

Or

$$z(u) \geq D_3z(u)C[u,v] \tag{3.7}$$

As in Case (a), we have

$0 \geq D_4z(t) + G(p)D_4z\big(\alpha(t)\big) + \lambda Q(t)G(D_3z(\beta(t)))G(C[\beta(t),\beta(s)])$

Integrating the above inequality, from $s$ to $\beta^{-1}(\alpha(s))$, we obtain

$$\int_s^{\beta^{-1}(\alpha(s))} \lambda Q(t)G(D_3z(\beta(t)))G(C[\beta(t),\beta(s)])\Delta t \leq D_3z(s) + G(p)D_3z(\alpha(s))$$

Implies

$$\lambda G(D_3z\big(\alpha(s)\big)) \int_s^{\beta^{-1}(\alpha(s))} Q(t)G(C[\beta(t),\beta(s)]))\Delta t \leq (1+G(p))D_3z\big(\alpha(s)\big) \leq (1+G(p))D_3z\big(\alpha(s)\big)$$

Or

$$\int_s^{\beta^{-1}(\alpha(s)} Q(t)G(C[\beta(t),\beta(s)])\Delta t \leq \frac{(1+G(p))D_3z\big(\alpha(s)\big)}{G(D_3z\big(\alpha(s)\big))} \leq \frac{(1+G(p))}{\lambda M_1}$$

Case $(c)$. For $u > s > v > t_2$,

$$D_2z(u) - D_2z(v) = \int_v^u D_3z(s)\Delta s \le (u-v)D_3z(v)$$

implies

$$D_2z(u) \le (u-v)D_3z(v)$$

or

$$r(u){D_1}^{\Delta}z(u) \le (u-v)D_3z(v)$$

or

$$-{D_1}^{\Delta}z(u) \ge -\frac{(u-v)}{r(u)}D_3z(v)$$

For $u > \theta > v > t_2$, we have

$$-D_1z(u) + D_1z(v) = \int_v^u \frac{(\theta - v)}{r(\theta)}(-D_3z(v))\Delta s,$$

implies

$$D_1z(v) \ge (-D_3z(v))\int_v^u \frac{(\theta - v)}{r(\theta)}\Delta\theta,$$

For $u > s > v > t_2$, we have

$$z(u) - z(v) \ge \int_v^u (-D_3z(s))\int_s^u \frac{(\theta - v)}{r(\theta)}\Delta\theta\,\Delta s,$$

Or

$$z(u) \ge (-D_3z(v))\int_v^u\int_s^u \frac{(\theta - v)}{r(\theta)}\Delta\theta\,\Delta s$$

Or

$$z(u) \ge (-D_3z(v))\int_v^u \frac{(\sigma(t)-v)(t-v)}{r(t)}\Delta t$$

Or

$$z(u) \ge (-D_3z(v))D[u,v] \tag{3.8}$$

As in Case (a), we have

$$\begin{aligned} 0 &\ge D_4z(t) + G(p)D_4z(\alpha(t)) + \lambda Q(t)G\left(z(\beta(t))\right) \\ &\ge D_4z(t) + G(p)D_4z(\alpha(t)) + \lambda Q(t)G\left(z(\gamma^{-1}(t))\right) \\ &\ge D_4z(t) + G(p)D_4z(\alpha(t)) + \lambda Q(t)G((-D_3z(\gamma^{-1}(s))))G(D[\gamma^{-1}(t),\gamma^{-1}(s)]) \end{aligned}$$

Integrating the above inequality from s to $\gamma^{-1}(s)$, we obtain

$$\int_s^{\gamma^{-1}(s)} \lambda Q(t)G(-D_3z(\gamma^{-1}(s)))G(D[\gamma^{-1}(t),\gamma^{-1}(s)])\Delta t \le -D_3z(\gamma^{-1}(s)) - G(p)D_3z(\gamma^{-1}(s))$$

implies

$$\begin{aligned} \lambda G(-D_3z(\gamma^{-1}(s)))\int_s^{\gamma^{-1}(s)} Q(t)G(D[\gamma^{-1}(t),\gamma^{-1}(s)])\Delta t &\le (1+G(p))(-D_3z(\gamma^{-1}(s))) \\ &\le (1+G(p))(-D_3z(\gamma^{-1}(s))) \end{aligned}$$

Or

$$\int_s^{\gamma^{-1}(s)} Q(t)G(D[\gamma^{-1}(t),\gamma^{-1}(s)])\Delta t \le \frac{(1+G(p))(-D_3z(\gamma^{-1}(s)))}{G(-D_3z(\gamma^{-1}(s)))} \le \frac{(1+G(p))}{\lambda M_1}$$

Case $(d)$. For $u > s > v > t_2$,

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$$D_2 z(u) - D_2 z(v) = \int_v^u D_3 z(s)\Delta s \geq (u - v) D_3 z(u)$$

implies

$$D_2 z(u) \geq (u - v) D_3 z(u)$$

or

$$r(u) {D_1}^{\Delta} z(u) \geq (u - v) D_3 z(u)$$

Or

$$D_1{}^{\Delta} z(u) \geq \frac{(u - v)}{r(u)} D_3 z(u)$$

For $u > \theta > v > t_2$, we have

$$D_1 z(u) - D_1 z(v) = \int_v^u \frac{(\theta - v)}{r(\theta)} (D_3 z(\theta))\Delta\theta,$$

Implies

$$-D_1 z(v) \geq D_3 z(u) \int_v^u \frac{(\theta - v)}{r(\theta)} \Delta\theta,$$

For $u > t > v > t_2$, we have

$$z(v) \geq \int_v^u (D_3 z(u)) \int_t^u \frac{(\theta - v)}{r(\theta)} \Delta\theta\, \Delta s,$$

Or

$$z(v) \geq D_3 z(u) \int_v^u \int_t^u \frac{(\theta - v)}{r(\theta)} \Delta\theta\, \Delta s$$

Or

$$z(v) \geq D_3 z(u) \int_v^u \frac{(\sigma(t) - v)(t - v)}{r(t)} \Delta t$$

Or

$$z(v) \geq D_3 z(u) D[u, v] \qquad (3.9)$$

As in Case (a), we have
$0 \geq D_4 z(t) + G(p) D_4 z\big(\alpha(t)\big) + \lambda Q(t) G(D_3 z(\beta(s))) G(D[\beta(s), \beta(t)]))$
Integrating the above inequality from $\beta(s)$ to $s$, we obtain

$$G(D_3 z\big(\beta(s)\big) \int_{\beta(s)}^{s} Q(t) G(D[\beta(s), \beta(t)])\Delta t \leq D_3 z\big(\beta(s)\big) + G(p) D_3 z(\alpha(\beta(s)))$$

implies
$\lambda G(D_3 z\big(\beta(s)\big) \int_{\beta(s)}^{s} Q(t) G(D[\beta(s), \beta(t)])\Delta t \leq (1 + G(p))(D_3 z(\beta(s)))$
Or

$$\int_{\beta(s)}^{s} Q(t) G(D[\beta(s), \beta(t)])\Delta t \leq \frac{(1 + G(p)) D_3 z\big(\beta(s)\big)}{G(D_3 z(\beta(s)))} \leq \frac{(1 + G(p))}{\lambda M_1}$$

If $z(t) < 0$ for $t \geq t_2$, then $z(t) = y(t) - k(t) < 0$. From this, $x(t) \leq y(t) = x(t) + p(t) x\big(\alpha(t)\big) < k(t)$. Since $k(t) > 0$ and $k^{\Delta}(t) < 0$, so $k(t)$ is bounded, it follows that, $x(t)$ is bounded. Which is a contradiction to that, $x(t)$ is unbounded. This completes the proof of the theorem.

**Theorem 3.2.** Let $-1 < p(t) \leq 0$ for $t \in [t_0, \infty)_T$ and $\beta(t) \leq \alpha(t))$, $\gamma^{-1}(t) \leq \beta(t)$ . Assume that, $(H_0{}')$, $(H_1)$, $(H_2)$, $(H_5)$ and $(\mathrm{H}_6)$ hold. Furthermore, assume that
$(H_{11})$ $\limsup_{t\to\infty} \int_s^{\beta^{-1}(\alpha(s))} g(t) G\big[B\big(\beta(t), \beta(s)\big)\big]\Delta t > \frac{1}{M_1}$;

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$(H_{12})$ $\limsup_{t\to\infty} \int_s^{\beta^{-1}(\alpha(s))} g(t)G[C(\beta(t),\beta(s))]\Delta t > \frac{1}{M_1}$;
$(H_{13})$ $\limsup_{t\to\infty} \int_s^{\gamma^{-1}(s)} g(t)G[D(\gamma^{-1}(t),\gamma^{-1}(s))]\Delta t > \frac{1}{M_1}$;
$(H_{14})$ $\limsup_{t\to\infty} \int_{\beta(s)}^{s} g(t)G[D(\beta(s),\beta(t))]\Delta t > \frac{1}{M_1}$
and $(H_{15})$ $\alpha^n(t) = \alpha(\alpha^{n-1}(t))$, $\lim_{n\to t} \alpha^n(t) = \infty$
hold. Then every unbounded solution of $(H)$ oscillates.

**Proof:** Suppose on the contrary that, $x(t)$ be an unbounded nonoscillatory solution of (H) such that $x(t) > 0$, $x(\alpha(t)) > 0$, $x(\beta(t)) > 0, x(\gamma(t)) > 0$ for all $t \geq t_1 > t_0$. The case $x(t) < 0$ can similarly be dealt with due to $(H_2)$. Proceeding as in Theorem 3.1, we get (3.4). Then there exists a $t_2 > t_1$ such that $D_i x(t)$ for $i = 0,1,2,3$ are eventually of one sign on $[t_2,\infty)_T$. Assume that, $z(t) > 0$ for $t \geq t_2$. Then $z(t) \leq x(t)$ for $t \geq t_2$ and hence (3.4 ) can be written as

$$D_4 z(t) + g(t)G(z(\beta(t))) \leq 0 \text{ , } t \geq t_2 \text{ .}$$

Ultimately, any one of four Cases $(a)-(d)$ of Lemma 2.2 holds for $t \geq t_2$. We can apply the arguments of the cases of Theorem 3.1, to get contradictions to $(H_{11})-(H_{14})$. Therefore, $z(t) < 0$ for $t \geq t_2$ and also $y(t) < k(t)$ for $t \geq t_2$ implies that $y(t)$ is bounded and hence $x(t)$ is bounded for $t \geq t_2$. Since $z(t)$ is monotonic, then $\lim_{t\to\infty} z(t)$ exists. On the otherhand, $x(t)$ is unbounded and therefore, it suggests that either $x(t) > x(\alpha(t))$ or $x(t) < x(\alpha(t))$ for $t \geq t_2$. If the former holds, then $z(t) = x(t) + p(t)x(\alpha(t)) - k(t) > (1 + p(t))x(\alpha(t)) - k(t) \to \infty$ as $t \to \infty$ gives a contradiction. If the later holds, then

$$x(t) < x(\alpha(t)) < x(\alpha^2(t)) < \cdots < x(\alpha^n(t)) < \cdots,$$

leads to the fact that $x(t)$ is bounded , which is absurd. Hence, the theorem is proved.

**Theorem 3.3.** Let $-\infty < p(t) < -1$ for $t \in [t_{0,}\infty)_T$. Assume that all conditions of Theorem 3.2 hold. If
$(H_{16})$ $\int_0^\infty g(u)du = \infty$
Then every unbounded solution of $(H)$ oscillates.

**Proof:** On the contrary, we proceed as in the proof of Theorem 3.2, and the case $z(t) > 0$ is similar. When $z(t) < 0$ for $t \geq t_2$, we have that, $y(t) < k(t)$ for $t \geq t_2$, so $y(t)$ is bounded and follows that, $z(t)$ is bounded. Because, $z(t)$ is monotonic, $\lim_{t\to\infty} z(t)$ exists. Here, we consider Cases $(b)-(f)$ of Lemma 2.2. For the Cases $(b),(d),(e)$; since $x(t)$ is unbounded, then either $x(t) > x(\alpha(t))$ or $x(t) < x(\alpha(t))$ for $t \geq t_2$. Suppose that, $x(t) < x(\alpha(t))$ for $t \geq t_2$. Then
$z(t) = x(t) + p(t)x(\alpha(t)) - k(t) < (1 + p(t))x(\alpha(t)) - k(t) \to -\infty$ as $t \to \infty$,
which gives a contradiction to the existence of the limit. Hence,

$$x(t) > x(\alpha(t)) > x(\alpha^2(t)) > \cdots > x(\alpha^n(t)) > \cdots,$$

that is, $\liminf_{t\to\infty} x(t) > 0$. Then there exists a $t_3 > t_2$ and $\eta > 0$ such that $x(\beta(t)) > \eta$ for $t \geq t_3$. Integrating (3.4 ) from $t_3$ to $\infty$, we get a contradiction to $(H_{16})$. In Cases $(c)$ and $(f)$, it is immediate to see that $\lim_{t\to\infty} D_1 z(t) = -\infty$ and $\lim_{t\to\infty} z(t) = -\infty$, so these cases are not possible. Hence, the proof of the theorem is complete.

**Theorem 3.4**. Assume that $0 \leq p(t) \leq p < \infty$ for $t \in [t_{0,}\infty)_T$. If $(H_0)$ and $(H_1)-(H_8)$ hold, then every unbounded solution of (H) oscillates.
**Proof:** The proof of the theorem follows from Lemma 2.1 and Theorem 3.1. Hence, the details are omitted.

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**Theorem 3.5.** Let $-1 < p(t) \leq 0$ for $t \in [t_{0,}\infty)_T$. If $(H_0)$, $(H_1)$, $(H_2)$, $(H_5)$, $(H_6)$, $(H_{11})$ and $(H_{12})$ hold, then every unbounded solution of (H ) oscillates.
**Proof.** The proof of the theorem follows from Lemma 2.1 and Theorem 3.2 and hence the details are omitted.

**Theorem 3.6.** Let $-\infty < p(t) < -1$ for $t \in [t_{0,}\infty)_T$. If all the conditions of Theorem 5 hold and $(H_{16})$ hold, then every unbounded solution of (H ) oscillates.
**Proof:** The proof of the theorem follows from Theorem 5 and Theorem 3 and hence the details are omitted.

***Remark 3.7.*** It would be interesting to apply the results of this work to the following nonlinear dynamic equations

$$(r(t)(x(t) + p(t)x(\alpha(t)))^{\Delta^2})^{\Delta^2} + \sum_{i=1}^{n}(-1)^{i+1}\, g_i(t)G_i\left(x(\beta_i(t))\right) = f(t),$$

for the cases

$$, f(t) = 0 \text{ for all } t \in [t_0, \infty)_T \quad \text{or } f(t) \neq 0 \text{ for } t \in [t_0, \infty)_T.$$

**Example 1**. Consider, $T = R$,

$$(e^{-t}(x(t) + (-\tfrac{1}{25}e^{\pi})x(t-\pi))'')'' + e^{\frac{3\pi}{2}}\left(\tfrac{52}{25}e^{-t} - e^{-3t}\right)x\left(t - \tfrac{3\pi}{2}\right) - e^{\frac{\pi}{2}}(e^{-2t} + e^{-\pi}cos^2t)e^{-t}\,\frac{x\left(t-\frac{\pi}{2}\right)}{1+x^2\left(t-\frac{\pi}{2}\right)} = 0\,,\ t \in [\tfrac{3\pi}{2}, \infty). \tag{3.10}$$

Equations (3.10), satisfies all the conditions of Theorem 5. Hence, every unbounded solution of (3.10) is oscillatory. In particular, $x(t) = e^t \sin t$ is an unbounded oscillatory solution of (3.10).

**Example 2**. Consider, $T = Z$,

$$\Delta^2\left(e^n\Delta^2\left(x(n) + (-1)x(n-2)\right)\right) + \{e(e+1)^2(e^2+1)^2e^n + 4e(e^{-1}+1)^2e^{-n} - \tfrac{e}{(n+1)e^{4n}}\}x(n-1) - e^2\left(\tfrac{e^4+e^{2n}}{(n+1)e^{4n}}\right)\frac{x(n-2)}{1+x^2(n-2)} = 0\,,\ n \geq 3. \tag{3.11}$$

Equations (3.11), satisfies all the conditions of Theorem 2. Hence, every unbounded solution of (3.11) is oscillatory. In particular, $x(n) = e^n \sin n$ is an unbounded oscillatory solution of (3.11).

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