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Involutions in Banach Algebra

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Abstract- In this paper, we define some definition related to involution, examples, Gelfand Nahnark Theorem has important role.

Keywords- Commutative Banach algebra with involuation Hermitian, maximal ideal space Gelfand transform, Runge's theorem in complex analysis.

I. INTRODUCTION

Involution has important role in Banach algebra. In this paper we discuss self adjoint, hermitian,Banach algebra corollary, Gelfand Nahnark Thoeorem, isometry and isomorphism studied

II. DEFINITION

A map $x \to x^*$ of a complex algebra A into A is called an invoution of A if it has the following properties for all $x, y \in A \text{and} \lambda \in C.$

(1) $(x+y)^* = x^*+y^*$

(2) $(\lambda x)^* = \bar{\lambda} x^*$

(3) $(xy)^* = y^* x^*$

(4) $x^{**} = x$

2.1 Definition :

If $x \in A \text{and} x^* = x,$ then *x* is called hermitian or self adjoint.

Example :

$f \to \bar{f}$ is an involution on C(X)

2.1 Theorem :

If A is a Banach algebra with an involution, and if $x \in A,$ then

a) $x+x^*, i(x-x^*) \text{and} xx^*$ are hermitian.

b) *x* has a unique representation x = u+iv where $u, v \in A$ and u and v are hermitian.

c) The unit element e is hermitian

d) *x* is invertible in A if and only if x* is invertible in which case $(x^*)^{-1} = (x^{-1})^*$ and

e) $\lambda \in \sigma(x) \text{iff} \bar{\lambda} \in \sigma(x^*)$

Proof:

a) $(x+x^*) = x^*+x^{**} = x^* x = x+x^*.$

Hence $x+x^*$ is hermitian.

$[i(x-x^*)]^* = \bar{i}(x-x^*)^* = -i[x^*-(x^*)^*] = -i(x^*-x) = i(x-x^*)$

$(xx^*)^* = (x^*)^* \cdot x^* = x \cdot x^*.$ Hence $i(x-x)^* \text{and} x^*$ and x* are hermitian.

b) Put $u = \frac{x+x^*}{2} \text{and} v = \frac{i(x^*-x)}{2}.$ Then x=u+iv. Clearly u, v are hermitian since $x+x^*$ is hermitian and also $\frac{i(x^*-x)}{2}$ is hermitian. The uniqueness of the representation is yet to be proved. If $u`+iv` = x$ is another representation then put $w = v`-v.$. Then both w and iw are hermitian and $iw = (iw)^* = -iw^* = -iw$ i.e. $iw+iw=0$ i.e. $2iw=0$ (ie) $v`=v.$. Since $v`=v, u`=u$

Hence the representation is unique.

c) Clearly $e^* = ee^*.$. But ee^* is self adjoint. Hence e* is self adjoint. Hence e is self adjoint.

d) Since x is invertible $\exists, x^{-1} s.t. x x^{-1} = e$

Now $(x x^{-1})^* = (x^{-1})x^* = e^* = e (\because$ e is self adjoint)

$\therefore (x^{-1})^*$ is the inverce of x*

But $(x^*)^{-1}$ is the inverse of x* and hence $(x^{-1}) = (x^*)^{-1}$

e) Let $\lambda \in \sigma(x)$. Then $(\lambda e - x)$ is not invertible. Hence $(\lambda e - x)^*$ is not invertible (ie) $(\bar{\lambda} e - x^*)$ is not invertible. Hence $\bar{\lambda} \in \sigma(x^*)$ the converse follows analogously.

3. Definition :

If A is a Banach algebra with an involution *, which satisfies the $\|xx^*\| = \|x\|^2$ for every $x \in A$ then A is called a B* algebra.

3.1 Theorem :

If A is a semi simple commulative Banach algebra, then involution on A is continuous

Proof:

Let h be a homomorphism of A

Define $Ø(x) = \bar{h}(x^*)$,

Then

$$Ø(x+y) = \bar{h}[x+y]^* = \bar{h}(x^* + y^*)$$

$$= \bar{h}(x^*) + \bar{h}(y^*) = Ø(x) + Ø(y)$$

$$Ø(\alpha.x) = \bar{h}[(\alpha, x)^*] = \bar{h}(\bar{\alpha} x^*)$$

$$= \overline{h(\bar{\alpha}, x^*)} = \overline{\bar{\alpha} h(x^*)}$$

$$= \alpha \cdot \bar{h}(x^*)$$

$$= \alpha \phi(x)$$

Similarly $Ø(xy) = \bar{h}((xy)^*) = \bar{h}(y^* x^*)$

$$= \overline{h(y^* x^*)}$$

$$= \overline{h(y^*) h(x^*)}$$

$$= \bar{h}(y^*) \bar{h}(x^*)$$

$$= Ø(y).Ø(x) = Ø(x).Ø(y)$$

Hence Ø is a complex homomorphism on A. Then Ø is continuous. For, suppose $x_n \to x$, and $x_n^* \to y$ in A

Then

$$\bar{h}(x^*) = Ø(x) = Lim Ø(x_n) = Lim \bar{h}(x_n^*) = \bar{h}(y)$$

This is true for every $h \in \Delta$.

Since A is semisimple x* = y. Hence $x \to x^*$ is continuous by closed graph theorem.

3.1 Corollary

A is a B* algebra, iff $\|x^*\| = \|x\| \forall x \in A$ and $\|xx^*\| = \|x\| \|x\|^*$

For, we have $\|x\|^2 = \|xx^*\| \le \|x\| \|x^*\|$

Hence $\|x\| \le \|x^*\|$ .... (1)

Similarly $\|x^*\| \le \|x^{**}\| = \|x\|$ .... (2)

From (1) and (2) $\|x^*\| = \|x\|$

Now $\|xx^*\| = \|x\|^2 = \|x\| \cdot \|x\| = \|x\| \|x^*\|$

Conversely, we have that if $\|x\| = \|x^*\|$ for every $x \in A$ and $\|xx^*\| = \|x\| \cdot \|x^*\|$, then $\|xx^*\| = \|x\| \|x^*\| = \|x\| \cdot \|x\| = \|x\|^2$. Hence A is a B*-Algebra

3.2 Theorem: Gelfand-Nahnark Theorem

Suppose A is a commulative B* algebra, with maximal ideal space Δ. The Gelfand transform is then an isometric isomorphism of A onto $C(\Delta)$ which has the additional property that

$$h(x^*) = \overline{h(x)} (x \in A, h \in \Delta)$$

or equivalently, that

$$(x^*)\hat{} = \bar{\hat{x}} (x \in A)$$

In particular, *x* is hermitian if and only if $\hat{x}$ is a real function.

The above theorem is called Gelfand - Nahnark theorem

Proof:

Let $u \in A s.t. u = u^*$. Let $h \in \Delta$. We have to prove that h(u) is real.

Put z=u+ite for real t. If $h(u) = \alpha + i\beta$ where α, β are reals then

$$h(z) = h(u + ite) = h(u) + h(ite)$$

$$= \alpha + i\beta + it.h(e)$$

$$= \alpha + i\beta + it = \qquad \alpha + i(\beta + t)$$

$zz^* = u^2 + t^2 e$ so that

$$\alpha^2 + (\beta + t)^2 = |h(z)|^2 \le \|z\|^2 = \|zz^*\| \le \|u\|^2 + t^2$$

or $\alpha^2 + \beta^2 + 2\beta t \le \|u\|^2 \quad \forall t \in$ Real

But this implies that $\beta = 0.$ = 0. Hence h(u) is real.

If $x \in A$, then $x = u + iv$ with $u = u^*, v = v^*$

Hence $x^* = u - iv.$. Since $\hat{u}$ and $\hat{\vartheta}$ are real, we have $(x^*)^\wedge(h) = \hat{u} - i\hat{v}](h)$ for every $h \in \Delta (i.e.)(x^*)^\wedge = \overline{\wedge}_x$

Thus $\overline{A}$ is closed under complex conjugation. By Stone Weierstrass theorem is dense in $C[\Delta]$

If $x \in A$ and $y = xx^*,$ then y = y*. Hence $\|y^2\| = \|y\|^2$. By induction, we get that llymi $\|y^m\| = \|y\|^m$ for every $m = 2^n.$

Hence $\|\hat{y}\|_\infty = \|y\|$ by the spectral radius formula. Since y = xx*

we have $\hat{y} = \hat{x}\widehat{(x^*)} = \hat{x}\,\overline{\hat{x}} = |\hat{x}|^2$

Hence $\|\hat{x}\|_\infty{}^2 = \|y\| = \|xx^*\| = \|x\|^2$ or $\|\hat{x}\|_\infty = \|x\|.$

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Thus $x \to \hat{x}$ is an isometry. Hence $\overline{A}$ is closed in $C(\Delta).$ Since A is also dense in $C(\Delta)$, we conclude that $\hat{A} = C(\Delta).$. Hence the proof.

3.3 Theorem :

If A is a commutative B* algebra which contains an element *x* such that the polynomials in *x* and *x** are dense in A. then the formula $(\Psi f)^\wedge = f\,o\,\hat{x}$ defines an isometric isomorphism Ψ of $C(\sigma(x))$ onto A which statisfies.

$\Psi\hat{f} = (\Psi f)^*$ for every $f \in C(\sigma(x)).$ More over if $f(\lambda) = \lambda$ on $\sigma(x)$ then $\Psi f = x.$

Proof:

Let Δ be the maximal ideal space of A. We know that $\hat{x}$ is a continuous function on Δ. The range of $\hat{x}$ is $\sigma(x).$ Suppose $h_1, h_2 \in \Delta$ and $\hat{x}(h_2),$ then $h_1(x) = h_2(x)$ and hence $h_1(x^*) = h_2(x^*)$ by the previous theorem. If P is any polynomial in x and x*, then $h_1(P) = h_2(P)$ since h_1, and h_2, are homomorphisms. By hypothesis, the elements of the form P(x,x*) are dense in A and since h_1, and h_2, are continuous we have $h_1(y) = h_2(y)$ for every $y \in A.$ Hence $h_1 = h_2$. Hence we have proved that $\hat{x}(h_1) = \hat{x}(h_2)$ for every $y \in A.$ Hence $h_1=h_2$, (i.e.) $\hat{x}$ is 1-1. Since $\hat{x}$ is continuous and onto $\sigma(x)$ and $|-|$ we have that *x* is homeomorphism of Δ onto $\sigma(x)$ (By Vadiyanatha swamy's theorem). The mapping $f \to f\,o\,\hat{x}$ is therefore an isometric isomorphism of $C(\sigma(x)) \to C(\Delta)$ which preserves complex conjugation.

By the previous theorem, each fox is the Gelfand tranform of a unique elements of A, which we denote by Ψf and which satisfies $\|\Psi f\| = \|f\|_\infty$ [Since we have that $(x^*)^\wedge = \hat{x}$ by the previous theorem]. We have $\Psi\hat{f} = (\Psi(f))^*.$. If $f(\lambda) = \lambda$, then $f\,o\,\hat{x} = \hat{x}$ so that we have $(\Psi\hat{f}) = \hat{x}(ie)\Psi f = x.$

We are interested in knowing the existence of square roots in a Banach algebra. The following theorem is one in that direction.

3.4 Theorem :

Suppose A is a commutative Banach algebra with an involution. If *x* is a self adjoint element of A and if $\sigma(x)$ contains no real number $\lambda \le 0$, then there exists $y \in A$ with $y^2 = x$ and y = y*.

Proof:

Let R denote the non positive real numbers and let $\Omega = C - R^-.$. There exists a holomorphic function $f \in H(\Omega)$ such that $f^2(\lambda)$ and f(1) = 1. Since $\sigma(x) \subset \Omega,$ we can define $y \in A$ as

$$y = \hat{f}(x) = \frac{1}{2\pi i}\int_\Gamma f(\lambda)(\lambda e - x)^{-1} d\lambda$$

Where Γ is any contour that surrounds $\sigma(x)$ in Ω . Then it can be proved that $y^2 = x$ [For a proof the student is referred to Defn and theorem of "Functional Analysis" by Rudin. This is the required y and y* = y. To prove y*=y we need what is called Runge's theorem in complex analysis.

Since Ω is simply connected 'Runges'. Theorem gives polynomials P, that converge to funiformly on compact subsets of Ω. Define Q_n, by $2Q_n(\lambda) = P_n(\lambda) + \overline{P_n(\lambda)}$. Since $f(\bar{\lambda}) = \overline{f(\lambda)}$ the polynomials $Q_n \to f$ in the same manner [(ie) uniformly on compact sets]

Define $y_n = Q_n(x).(n = 1, 2, 3, ...)$ By definition, the polynomials Q_n have real coefficients. Since x = x*, if follows that y_n, = y_n*

The element $y = \lim_{n\to\infty} y_n$, and hence y = y*. if f* is continuous. Even if f* is not assumed to be continuous we can give a different argument to prove that y=y* as follows.

Let R be the radical of A. Let $\pi : A \to A \setminus R$ be the quoteint map. Define an involution in A/R by

$[\pi(a)]^* = \pi(a^*) \text{ for } a \in A$

If a is heremitian, then so is it $\pi(a)$

Since π it is continuous, $\pi(y_n) \to \pi(y)$

Since A/R is isomorphic to A. A/R is semi simple and therefore every involution in A/R is continuous. Hence $\pi(y)$ is hermitian. Hence $\pi(y - y^*) = 0 \text{(ie)} y - y^*$ is in the radical of A.

Now we can write y = u + iv where u and v are hermitian. Since $y - y^* \in R,$ hermitian v belongs to the radical of A. Since x = y^2 we have $x = u^2 - v^2 + 2iuv.$.

Let h be a complex homomorphism on A. Since v is in the radical of A, h(v) = 0. Hence $h(x) = [h(u)]^2$

By luypothesis $0 \notin \sigma(x).$ Hence $h(x) \neq 0$. Hence $h(x) \neq 0$. This is true for every $h \in \Delta \text{(ie)} u$ is invertible.

Since x=x*

and since $x = u^2 - v^2 + 2iuv,$ we have that uy = 0

Since v=u^{-1} (u v) we have that v = u^{-1}.0 = 0

Hence v = u and hence y* = u* But u is hermitian and hence y* = y

III. CONCLUSION

It is clear that A is semi simple commutative Banach algebra with an involution which satisfy involution on A is continuous, Gelfand Nahnork theorem gives isometric isomorphism of A onto A which satisfy certain condition and Runges theorem give polynomials P, that converge to uniformly on compact subjects.

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