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Role of Ideals and Homomorphisms in Banach Algebra

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Abstract- ***In thepresent paper we discuss ideal, maximal ideal, theorem homomorphism.***

Keywords- Ideal in commutative Banach algebra, maximal proper ideal, AIJ complex, algebra, complex homeomorphisms.

I. INTRODUCTION

Commutative Banach algebras make an interesting reading. The Banach algebras have a nice theory in themselves. The ones that occur to our mind in a natural way are the Banach algebras $C[a,b] \text{or} C[X]$ C (a,b) or C[X] which denote the continuous real (complex) valued functions on [a,b] or a compact T2 space X respy. The theory of Gelfand looks very natural

II. DEFINITION

A subset J of a commutative Banach algebra A is said to be an ideal if

1) J is a subspace of A (as a vector space) and
2) $xy \in J$ if $x \in A$ and $y \in J$

If $J \neq A,$ then J is called a proper ideal.

Ex.:

Let $A = C[0,1].$ Let

$$J = \{ f \in C[0,1] / f(0) = 0 \}$$

It is easy to check that is an ideal.
In general if E c [0,1],
consider

$$J_E = \{ f \in C[0,1] / f(E) = 0 \}$$ J,

Clearly if $f, g \in J_E$ then $f + g \in J_E$.

Also iff $f \in J_E$ and $g \in A,$ then consider $g\ f(E)$

$$g\ f(E) = \{ g, f(x) | x \in E \} = \{ g(x) \cdot f(x) / x \in E \} =$$

Hence $g\ f \in J_E$.

Therefore J_E), is an ideal in C[0, 1]

2.1 Definition :

An ideal $J \subset A,$ where A is a commutative Banach algebra is said to be a maximal ideal if J is not contained in any larger proper ideal.

Remark :

Every Commutative Banach algebra A with identity e contains a maximal ideal. For let $\mathfrak{I}$ denote the set of all proper ideas of A. $\mathfrak{I}$ is partially ordered by set inclusion Let $A_1 \subset A_2 \subset \ldots$ be any chain (Totally ordered sub collection) of $\mathfrak{I}.$ Then each Ai is a proper ideal of A and $\bigcup_{i=1}^{\infty} A_1$ is also a proper ideal. For $e \notin Ai$ for any i and hence $e \notin \cup Ai.$ Hence any chain in $\mathfrak{I}$ has an upper bound. Hence by Zorn's Lemma, there exists a maxi element in $\mathfrak{I}$. But elements of $\mathfrak{I}$ are proper ideals and hence $\mathfrak{I}$ has a maximal proper a for A.

2.1 Theorem:

Any proper ideal J in a Cummulative Banach algebra A with unit element contain any invertible clement.

Proof:

Let u be an invertible element. Then u^{-1} exists. If J contains u. then $uu^{-1} \in J$ for $u \in J$ and $u^{-1} \in A$ (Since J is an ideal)

(i.e.) $e \in J$

Hence if $x \in A$, then $xe \in J$, (ie) $x \in J$.

$\therefore A \subset J$. This is contradiction to the fact that J is a proper ideal of A. Hence the result.

2.2 Theorem :

It is an ideal in a commutative Banach algebra, then $\bar{J}$ is also an ideal.

Proof:

Let $x \in \bar{J}$ and $y = A$.. Since $x \in \bar{J}$ there exists a sequence of elements $x_n \in J$ such that $x_n \to x$. . Clearly $x_n y \to xy$ as multiplication in A is continuous. Further $x_n y \in J \forall n$ is an ideal.

Hence $xy \in \bar{J}$.. Therefore $\bar{J}$ is an ideal.

2.3 Theorem :

(1) If A is a commutarive Banach algebra with unit e, then every proper ideal of A contained in a maximal proper ideal of A.

(2) If A is a commutative Banach Algebra, then every maximal ideal is closed.

Proof:

Let J be a proper ideal of A. Let $\Im$ = {The set of all proper ideals of A which com J) Then $\Im = \theta$ since $J \in \Im$. Partially order $\Im$ by inclusion (ie) Define $J_1 \geq J_2$ if $J_1 \supset J_2$. Then we can apply Hausdort maximalty principle for $\Im$. Let L be a maximal total ordered sub collection of $\Im$. Let I be the Union of members of L Clearly I is an ideal forif $x, y \in I$, then $x \in D_1$ and $y \in D_2$ for some $D_1, D_2 \in L$. Since L is totally ordered either $D_1 \subset D_2$ or $D_2 \subset D_1$. Without loss of generality let us assume that $D_1 \subset D_2$. Then $x, y \in D_2$. Since D_2, is an ideal $x + y \in D_2$. Hence $x + y \in I$. Similarly if $x \in I$ and $y \in A$, then $x \in D$ for some $D \in L$. But D is an ideal therefore $xy \in D$.. Hence $xy \in I$. Therefore I is an ideal Obviously $J \subset I$ and $I \neq A$ since no member of $\Im$ contains the unit element and $c \notin I$. This maximality of L implies that I is a maximal ideal. For if M is containing proper ideal such that $M \supset J$. then, since $M \supset J, M \in \Im$ and $M \notin L$. Hence $L \cup \{M\}$ is a bigger chain than contradicting the maximality of L.

(2) Suppose M is a maximal ideal of A. Then M does not contain any invertible elements. But the set G of invertible elements is an open set. Hence $M \cap G = \theta$. Hence $M \subset A - G$. Hence $\bar{M}$ does not contain any invertible element. Hence $\bar{M}$ is a proper ideal on A. But M is a maximal proper ideal and hence $\bar{M} = M$.$\therefore$ =M is closed.

Now let us look at the remark that we have made, namely it is a proper ideal, so $\bar{J}$. Clearly $\bar{J}$ is an ideal. It remains to be seen that $\bar{J}$ is proper. Since is proper ideal J is contained in a proper maximal ideal M. But M is closed. Hence $\bar{J} \subset M$. Hence $\bar{J}$ is proper

III. METHODOLOGY

3.1 Definition :

Let A and B be commutative Banach algebras over C Let $\emptyset : A \to B$.

Ø is said to be a homomorphism if

Ø (x+y) = (x)+Ø(y), for all Ø

$\emptyset(\alpha x) = \alpha \emptyset(x)$, $for\, x \in A, \alpha \in C$.

$\emptyset(xy) = \emptyset(x)\emptyset(y) \forall x, y \in A$

Let N be the null space of $\emptyset$

Then $N = \{x \in A / \emptyset(x) = 0\}$. Now N is an ideal in A. Since Ø is linear. N is clearly subspace. But,if $x \in N$ and $y \in A$, then $\emptyset(xy) = \emptyset(x)\emptyset(y) = 0$.

Hence $xy \in N$. Consequently N is an ideal in A. Now we can see that if $\emptyset$ is continuous, then $N = \emptyset^{-1}\{0\}$ and since {o} is closed in B, N is a closed ideal in A.

Suppose J is a proper closed ideal in A and $\pi : A \to A/J$ is the quotient map given by $x\,y \in N.$. then A/J is a Banach space with $\|x+y\|$ defined as Inf $\{\|x+y\| : y \in J\}.$

We can define a multiplication on AJ and make it into an algebra Further the nom defined above on A/J makes it into a Banach algebra

The map $\pi : A \to A/J$ is a homomorphism.

The multiplication in A/J is defined as (x+ y)(y+J)=xy+J. This is a well defined product on A/J for the following reason. If x+J=x^1+J then x-$x^1 \in$J. Similarly if y+J=y^1+J. then y-$y^1 \in$ J.

We claim that

(x+J) (y+J) = (x^1+J)(y^1+J)

(i e) $xy + J = x^1y^1 + J.$ This is true if and only if $xy - x^1y^1 \in J.$. Hence it and only if $x^1y^1 - xy \in J.$

But we have the following identity namely

$$(x^1y^1 - xy) = (x^1 - x)y^1 + x(y^1 - y)$$

Since $x^1 - x \in J \,\text{and}\, y^1 - y \in J,$ we see that the right side of the above equation is in J and consequently the left side of the above equation is in J. Hence the left side is an element of Hence the multiplication is well defined.

It can be now easily checked that A/J is a complex algebra. For we have $(x+J)[(y+J)(z+J)]$

$$= (x+J)[yz+J]$$
$$= x(yz)+J$$
$$= (xy)z+J$$
$$= (xy+J)(z+J)$$
$$= [(x+J)(y+J)](z+J)$$

Hence the product is associative. Similarly we can prove the other requirements of algebra and hence A/J is a complex Banach algebra.

Now $\pi : A \to A/J$ is the usual quotient map.

Since $\|\pi(x)\| \le \|x\|,$ by the definition of norm on A/J we get that π is continuous. Further we have that if $x_1, x_2 \in A \,\text{and}\, \delta > 0$ then $\|x_i + y_i\| \le \|\pi(x_i)\| + \delta$ for some $y_i \in J\,(i = 1,2)$

Since $(x_1 + y_1)(x_2 + y_2) \in x_1x_2 + J$

we have

$$\|\pi(x_1 \cdot x_2)\| \le \|\pi(x_1 + y_1)x_2 + y_2)\| \le \|x_1 + y_1\| \cdot \|x_2 + y_2\|$$

so that

$$\|\pi(x_1)\pi(x_2)\| \le \|\pi(x_1)\| \cdot \|\pi(x_2)\| \qquad (*)$$

Since π it is an onto map. we have $\|z_1z_2\| \le \|z_1\|\|z_2\|$ in A/J.

Further if e is the identity of A, then $\pi(e)$ is the identity of A/1.

But $\pi(e) = e + J \ne J$ and hence $\pi(e) \ne 0$

Since $\|\pi(x)\| \le \|x\|$ for every *x*. we have that

$$\|\pi(e)\| \le \|e\| = 1. \qquad (*)(*)$$

But we have

$$\|\pi(e)\pi(e)\pi(e)\| \le \|\pi(e)\| \cdot \|\pi(e)\| \,\text{from}\, (*)$$

(i e) $\|\pi(e^2)\| \le \|\pi(e)\|^2$

(i e) $\|\pi(e)\| \le \|\pi(e)\|^2$

(i e) $\|\pi(e)\| \ge 1.$

By combining with (*) (*) we get that $\|\pi(e)\| = 1$

$\therefore \pi(e)$ is the identity of A/J

$\therefore$ A/J is a Banach algebra.

Remark :

As has been remarked earlier, any complex nonzero homomorphism of $A \to C$ is called a multiplicative linear functional. These multiplicative linear functionals (complex homomorphisms) play an important role in the study of the Banach algebras.

We now consider the set A of all complex homomorphisms of A. We now give a topology on A and make it into a compact T_2, space. Each element of A will be viewed as a continuous function on Δ and hence A will be viewed as a subset of $C[\Delta]=$ set of all continuous complex functions on Δ. One will be naturally tempted to ask wherther $A=C|\Delta|$? If not what conditions on A will make it equal to C(Δ)?

3.1 Theorem:

Let A be a commutative Banach algebra with e. Let Δ be the set of all complex homomorphisms of A then every maximal ideal of A is the kernel of some $h\in\Delta$.

Proof:

Let M be a maximal ideal of A. Then we know that M is closed in A. Hence A/M is a Banach algebra. Choose $x\in A-M$.

Let $J=\{ax+y|a\in A \text{ and } y\in M\}$

Then $x\in J$. Also J is an ideal. I clearly contains M and hence J strictly contains M as $x\in J-M$. This forces J to be equal to A. Since M is the maximal ideal in A.

Hence $ax+y=e$ for some $a\in A, y\in M$

If $\pi : A\to A/M$ is the quotient map, we have $\pi(a)\pi(x)=\pi(e).$ Hence every non zero elements $\pi(x)$ of the Banach algebra A/M has an inverse in A/M. By Gelfand-Mazur theorem, there exists an isomorphism Ø:A/M$\to$C. Put $\text{h}=\text{Ø o}\pi.$

Then $h: A\to C$ and since both π and Ø it and are homomorphism h is a homomorphism of $A\to C.$ The null space of this homomorphism is clearly M. Hence we have he A whose null space is M.

3.2 Theorem:

Let A be a commutative Banach algebra with e and let Δ be the set of all complex homomorphisms on A. If $h\in\Delta$ then kernel h is a maximal ideal of A.

Proof:

Clearly ker h = Null space of his an ideal of A. Algebraically (A/ker h) is isomorphic to complex numbers. Hence ker h is a maximal ideal. For if $M\underset{\neq}{\supset}Ker\, h$ is an ideal and $\text{Ø}\cdot\frac{\text{A}}{\text{Ker h}}\to\text{C}$ given by $x+\text{Ker h}\to x \text{ then Ø(M)}$ is an ideal in C. But since C has no proper ideals, either $\text{Ø}^{-1}(\text{Ø(M)})=\text{M}$ is the whole of A or zero.

Hence Ker h is clearly a maximal ideal.

3.3 Theorem:

Let A be a commutative Banach algebra and Δ denote the set of all complex homomorphism on A. An element $x\in A$ is invertible in A if and only if $h(x)\neq 0$ for every $\text{h}\in\Delta.$

Proof:

Let $x\in A$ be invertible. Then $\exists x^{-1}\in A.$ If h is a complex homomorphism then $h(x,x^{-1})=h(e)=1=h(x)h(x^{-1}).$ Hence $h(x)\neq 0.$ Conversely if $h(x)\neq 0$ for any $h\in\Delta$, then $x\notin$ any maximal ideal of A.

Suppose *x* be not invertible. Then $I=\{ax|a\in A\}$ is an ideal of A.

But this ideal is contained in a maximal ideal M and hence there exist a complex homoorphism $h\in\Delta$ such that $h(M)=0$ and hence $h(x)\neq 0$: contradiction. $\therefore x\in A$ is invertible.

3.4 Theorem:

Let A be a commutative Banach algebra and let Δ be the set of all complex homomorphisms of A. An element $x\in A$ is invertible if and only if *x* lies in no proper ideal of A.

Proof :

If *x* lies in no proper ideal of A, then *x* does not lie in any maximal ideal. Hence, for so $h\in\Delta, h(x)=0.$. Hence by previous theorem *x* is invertible. Conversely if *x* is invertible and $x\in I,$ for a proper ideal I. then since $x^{-1}\in A, x.x^{-1}\in I\Rightarrow e\in I$ and hence I=A. which contradicts the fact I is a proper ideal. Hence the theorem.

3.5 Theorem:

Let A be a commutative Banach algebra and Let A denote the set of all complex homomorphisms on A. $\lambda \in \sigma(x)$ if and only if $h(x) = \lambda$ for some $h \in \Delta.$.

Let $\lambda \in \sigma(x).$. Thus $(x - \lambda e)$ is not invertible. Hence for some $h \in \Delta, h(x - \lambda e) = 0$ (ie) $h(x) = \lambda h(e) = \lambda.$ Conversely if $h(x) = \lambda$ for some h, then $h(x - \lambda e) = 0.$ Hence $(x - \lambda e)$ belongs to the null space of h which is a maximal ideal. Therefore by the above theorem $(x - \lambda e)$ is not invertible and hence $\lambda \in \sigma(x)$.

Examples

Find the maximal ideals of $C[0,1]$. It is an exercise for the student to prove that for any $x_0 \in [0,1], \text{if } h_{x0} : C[0,1] \to C$ is given by $h_{x0}(f) = f(x_0),$ then h is a complex any $C[0,1]$ its kernel is $M_{x0} = \{ f \in C[0,1] / f(x_0) = 0 \}$ By the theorems we have proved M_{x0} is a maximal ideal of $C[0,1]$. It is interesting to note that any maximal ideal of $C[0,1]$ occurs in this form. Further if $x_0 \neq y_0$ are elements of [0,1] then $M_{x0} \neq M_{y0}.$. Since by Urysohn's lemma we can always find a continuous function on [0,1] which vanishes at x_0 but not at $y_0.$. Hence we find a one-one correspondence between points of [0,1] and the points of $\Delta.$

II. CONCLUSION

It is clear that ideal is contained maximal ideal M and there exist complex homomorphism. Every element in commutative Banach algebra A is invertible if it is no proper ideal of A. It is there is relation between commutative clear that Banach algebra, complex homomorphism and maximal ideal.

References

1. Strichartz, Robert (2000). The Way of Analysis. Jones and Bartlett. ISBN 0-7637-1497-6.
2. Folland, Gerald B. (1999). Real Analysis: Modern Techniques and their Applications. Wiley. ISBN 0-471-31716-0.
3. Dudley, R. M. (2002). Real Analysis and Probability (2 ed.). Cambridge University Press. ISBN 0-521-00754-2.
4. Carothers, N. L. (2000). Real Analysis. Cambridge University Press. ISBN 0-521-49756-6.
5. Royden, H. L. (1988). Real Analysis. Prentice Hall. ISBN 0-02-404151-3.