

Modelling The Loaded Membrane of The Mridangam: A Finite-Element Study Inspired By Raman's Acoustics, With Implications for Standardised Drumhead Manufacturing

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Abstract- In 1920 and again in 1934 C. V. Raman showed that the right-hand head of the mridangam and the tabla, which carries a central loaded patch, produces overtones that are nearly harmonic, unlike an ordinary drum. He identified nine normal modes that group into the first five harmonics. This paper revisits Raman's result with the finite-element method and asks what it implies for the standardised manufacture of drumheads. The head is modelled as an ideal circular membrane under uniform tension with a concentric patch of higher areal density. An axisymmetric solver using quadratic elements, applied to each angular order, and an independent two-dimensional solver using linear triangles are validated against the exact Bessel-function frequencies and agree with each other to within 0.13%. Optimising the patch radius and density ratio to minimise the root-mean-square (RMS) deviation of Raman's nine modes from a harmonic series gives a radius ratio of 0.414 and a density ratio of 8.9, which reduces the inharmonicity from 216 cents for a uniform membrane to 37 cents. A two-zone patch, thicker at the centre, reduces it further to 15 cents. Sensitivity and Monte Carlo analyses show that harmonicity tolerates errors of $\pm 5\%$ in patch mass, is more sensitive to patch radius, and is almost unaffected by offsets of the patch up to 10% of the radius. These results translate into quantitative tolerances for a manufacturing specification.

Keywords: mridangam; tabla; C. V. Raman; loaded membrane; finite-element method; harmonic overtones; tolerance analysis; musical instrument manufacturing

I. INTRODUCTION

Most drums do not produce a clear musical pitch. The natural frequencies of a uniform circular membrane are proportional to the zeros of Bessel functions and do not form a harmonic series, so the sound of a struck membrane lacks the integer frequency ratios that the ear associates with a definite note [1], [2]. The Indian drums mridangam and tabla are a celebrated exception. Their right-hand heads carry a black circular patch, known as the *karanai* on the mridangam and the *syahi* on the tabla, built up from a paste containing iron oxide, starch and gum, and these heads sound with a clear pitch.

C. V. Raman studied these instruments as part of a long programme of research on musical acoustics that he pursued at the Indian Association for the Cultivation of Science in Calcutta from about 1907, alongside his better-known work in optics [3], [4]. That programme included a detailed mechanical theory of the bowed string, tested with a mechanically operated violin [5], and studies of Indian stringed and percussion instruments. In a short note with Sivakali Kumar in 1920 and in a fuller paper in 1934, he reported that the loaded head has nine low-frequency normal modes that fall into five groups, with frequencies close to the ratios 1:2:3:4:5 [6], [7]. The fundamental is the axisymmetric mode (0,1); the second harmonic is the mode (1,1) with one nodal diameter; the third harmonic is shared by the modes (2,1) and (0,2); the fourth by (3,1) and (1,2); and the fifth by (4,1), (2,2) and (0,3). Here (m, n) denotes m nodal diameters and n nodal circles,

counting the rim. Later analyses treated the head as a composite membrane [8], measured the vibrations of loaded drums [9], and examined the problem mathematically and numerically [10], [11].

The loaded drumhead is also a classic example of an inverse problem in acoustics. Kac's question "Can one hear the shape of a drum?" asked whether the spectrum of a membrane determines its shape [12]. The Indian drum-maker faces the converse problem: how to shape the distribution of mass so that the spectrum takes a prescribed form. Raman's analysis showed that the traditional design solves this problem remarkably well, but it left open how precisely the patch must be made, which is the question that matters for production.

The present study has two aims. The first is to reproduce Raman's result with the finite-element method, a general tool that can handle arbitrary patch shapes, and to quantify how closely an idealised loaded membrane can approach a harmonic series. The second is practical. Drumheads are still made largely by hand, and the quality of an instrument depends heavily on the skill of the maker. If the relationship between patch geometry and harmonicity is known quantitatively, it becomes possible to state manufacturing tolerances, which is the first step towards standardised production. Section 2 describes the model, Section 3 the numerical methods and their validation, Section 4 the results, and Section 5 the implications for manufacture.

II MATHEMATICAL MODEL

1. Governing equation

The drumhead is idealised as a thin circular membrane of radius a , fixed at its rim and stretched under a uniform tension T per unit length. Its areal density is σ_0 outside a concentric patch of radius b and $\sigma_p = \alpha\sigma_0$ inside it. Bending stiffness of the patch and the loading of the membrane by the surrounding air are neglected; their effects are discussed in Section 6. Small transverse displacements $w(r, \theta, t)$ satisfy [2], [13]

$$T \nabla^2 w = \sigma(r) \frac{\partial^2 w}{\partial t^2}, \quad w(a, \theta, t) = 0$$

with

$$\sigma(r) = \begin{cases} \alpha \sigma_0, & r < b \\ \sigma_0, & b \leq r \leq a \end{cases}$$

Fig. 1 shows the geometry. The two dimensionless parameters are the radius ratio $\beta = b/a$ and the density ratio α .

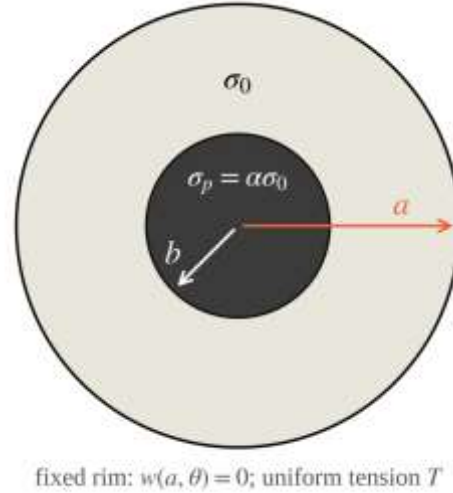


Fig. 1. Model of the loaded drumhead: a circular membrane of radius a and areal density σ_0 with a concentric patch of radius b and areal density $\alpha\sigma_0$.

2. Separation of variables

Because the density depends only on r when the patch is concentric, solutions may be sought in the form $w = R(r)\cos(m\theta)e^{i\omega t}$, which gives the radial equation

$$-\frac{T}{r} \frac{d}{dr} \left(r \frac{dR}{dr} \right) + \frac{Tm^2}{r^2} R = \omega^2 \sigma(r) R, \quad R(a) = 0$$

For a uniform membrane ($\alpha = 1$) the solutions are Bessel functions, and the natural frequencies are

$$\omega_{mn} = \frac{j_{mn}}{a} \sqrt{\frac{T}{\sigma_0}}$$

where j_{mn} is the n th zero of J_m [14]. The ratios j_{mn}/j_{01} are 1, 1.59, 2.14, 2.30, 2.65, 2.92, ..., far from harmonic. For a loaded membrane the solution is piecewise a combination of Bessel functions, which can be matched at $r = b$ [8]; the finite-element method avoids the matching and extends naturally to patches that are not concentric. (1)

3. Scaling and tuning

Equation (1) contains only the ratio T/σ_0 and the radius a , so every frequency of a given design scales as

$$f = \frac{\omega}{2\pi a} \sqrt{\frac{T}{\sigma_0}}, \quad \frac{\delta f}{f} = \frac{1}{2} \frac{\delta T}{T}$$

The player tunes the instrument by changing the tension through the leather straps and the braided rim, which shifts all frequencies together without changing their ratios [13]. A manufacturing error that changes only the overall pitch can therefore be corrected in use, whereas an error that changes the frequency ratios cannot. This distinction underlies the analysis of tolerances in Section 4.4. A change of 1% in tension shifts the pitch by about 8.6 cents, so an error of 34 cents requires a change of tension of about 4%.

4. Measure of harmonicity

Let f_i , $i = 1, \dots, 9$, be the frequencies of Raman's nine modes and $k_i \in \{1, 2, 3, 4, 4, 5, 5, 5\}$ the harmonic numbers he assigned to them. The deviation of each mode from a harmonic series with base frequency f_h is expressed in cents, a logarithmic unit equal to one hundredth of an equal-tempered semitone:

$$c_i = 1200 \log_2 \frac{f_i}{k_i f_h}, \quad \log_2 f_h = \frac{1}{9} \sum_{i=1}^9 \log_2 \frac{f_i}{k_i}$$

The base frequency chosen in this way minimises $\sum c_i^2$, and the root-mean-square deviation

$$\text{RMS} = \sqrt{\frac{1}{9} \sum_{i=1}^9 c_i^2}$$

is used as the measure of inharmonicity. A trained listener can detect mistuning of a few cents in sustained tones, but the overtones of a struck drum decay quickly, so deviations of the order of 10–20 cents are of little musical consequence [13], [15].

3. FINITE-ELEMENT METHODS AND VALIDATION

1. Axisymmetric solver

For each angular order m , Eq. (3) is multiplied by a test function $v(r)$ and integrated by parts to give the weak form

$$\int_0^a T \left(R'v' + \frac{m^2}{r^2} Rv \right) r dr = \omega^2 \int_0^a \sigma(r) R v r dr$$

The interval $[0, a]$ is divided into quadratic elements with a node placed exactly at $r = b$, so that the density is constant within each element. Integrals are evaluated by five-point Gauss–Legendre quadrature, which avoids the singular point $r = 0$. Assembly gives the generalised eigenvalue problem

$$\mathbf{K} \mathbf{u} = \omega^2 \mathbf{M} \mathbf{u}$$

with the condition $R(a) = 0$ for all m and, in addition, $R(0) = 0$ for $m \geq 1$ [16], [17]. The symmetric matrices are small and the problem is solved with standard dense routines [18].

2. Two-dimensional solver

To treat patches that are not concentric, and to visualise mode shapes, an independent two-dimensional solver was written. The disk is triangulated with nodes on concentric rings and on the boundary of the patch, and linear triangular elements are used. For a triangle of area A_e with nodal coordinate differences b_j and c_j , the element matrices are [19]

$$\mathbf{K}_e = \frac{T}{4A_e} (\mathbf{b}\mathbf{b}^T + \mathbf{c}\mathbf{c}^T), \quad \mathbf{M}_e = \frac{\sigma_e A_e}{12} \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

where σ_e is the density at the centroid of the element. The lowest 12–24 eigenpairs of the sparse system are found by shift-and-invert iteration.

3. Validation

All calculations are reported in non-dimensional form with $a = 1$, $T = 1$ and $\sigma_0 = 1$; physical frequencies follow from $f = \omega \sqrt{T/\sigma_0} / (2\pi a)$. Fig. 2 shows the maximum relative error in the first three frequencies of orders $m = 0-4$ for a uniform membrane, compared with Eq. (4). The error of the axisymmetric solver falls from 1.8×10^{-2} with elements of size $0.25a$ to $4.4 \times$

10^{-8} with elements of size $0.008a$, a rate of convergence close to the fourth power of element size expected for quadratic elements.

The two-dimensional solver converges at the second-order rate expected for linear triangles, with a maximum error of 2.4×10^{-4} at element size $0.01a$. For the optimised loaded head of Section 4, the two solvers agree to within 0.13% for all nine of Raman's modes. The axisymmetric solver with 30–40 elements per zone was used for all parameter sweeps.

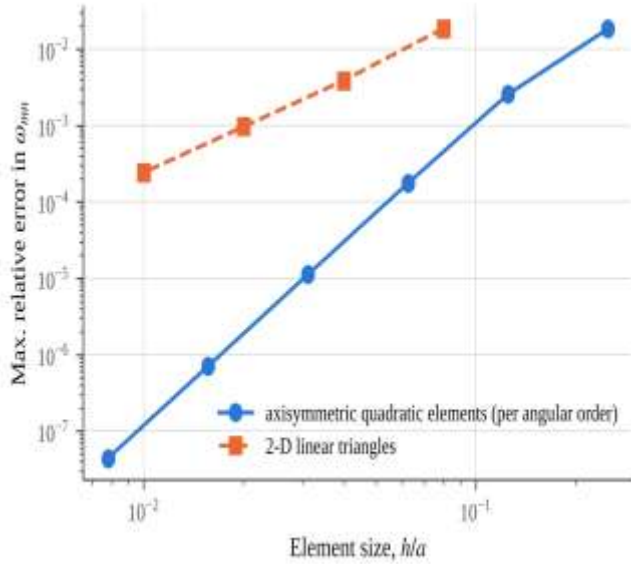


Fig. 2. Convergence of the two finite-element solvers for a uniform membrane: maximum relative error in ω_{mn} ($m = 0-4$, $n = 1-3$) against element size.

IV. RESULTS

1. The uniform membrane

For a uniform membrane, Raman's nine modes deviate from the best-fitting harmonic series by an RMS of 216 cents, more than two semitones. This confirms quantitatively that an unloaded head cannot sound a clear pitch.

2. Optimal single-zone patch

Fig. 3 maps the RMS inharmonicity of Eq. (7) over radius ratios $0.2 \leq \beta \leq 0.75$ and density ratios $1 \leq \alpha \leq 40$. A single basin of low inharmonicity lies at intermediate

radius and moderate loading. Optimisation from the best grid point gives

$$\beta^* = 0.414, \quad \alpha^* = 8.91, \quad \text{RMS} = 37.1 \text{ cents}$$

The patch then adds 1.36 times the mass of the bare membrane. For fixed radius ratios, the best density ratio and the resulting inharmonicity are $\alpha = 23.3$ and 83 cents for $\beta = 0.3$, $\alpha = 9.8$ and 38 cents for $\beta = 0.4$, and $\alpha = 5.8$ and 65 cents for $\beta = 0.5$. For $\beta = 0.6$ no finite optimum exists; the inharmonicity decreases slowly towards 109 cents as the patch becomes infinitely heavy. Harmonicity therefore requires a patch whose radius is about four tenths of the head radius.

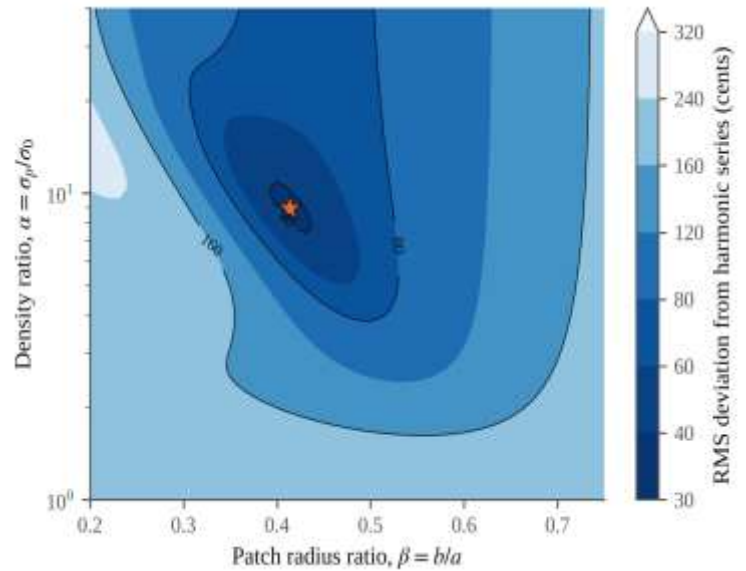


Fig. 3. RMS deviation of Raman's nine modes from a harmonic series as a function of patch radius ratio β and density ratio α . The star marks the optimum of Eq. (11).

Fig. 4 shows how the frequency ratios f_{mn}/f_{01} develop as the density ratio increases at $\beta = \beta^*$. At $\alpha = 1$ the modes are spread irregularly. As the patch becomes heavier the modes that move the patch strongly fall in frequency relative to those that do not, and the nine modes gather into the five groups that Raman identified. At the optimum the ratios to the best-fitting base frequency are 1.016, 1.947, 2.949 and 3.062, 3.965 and 4.103, and 4.964, 5.141 and 4.855.

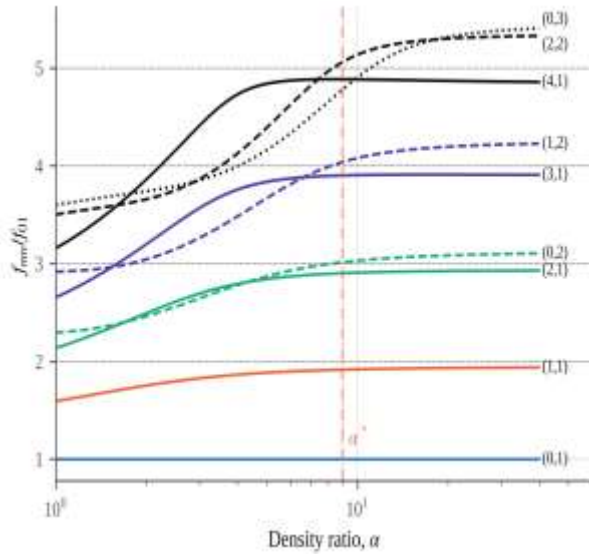


Fig. 4. Frequency ratios f_{mn}/f_{01} of Raman's nine modes as functions of the density ratio at $\beta = 0.414$. Dotted horizontal lines mark the harmonics 2–5; the vertical line marks α^* .

Fig. 5 shows the mode shapes computed by the two-dimensional solver at the optimum, labelled with their ratios to the harmonic base frequency. In the modes with nodal circles, (0,2), (1,2), (2,2) and (0,3), one nodal circle lies close to the edge of the patch, which is why their frequencies are so sensitive to the patch radius.

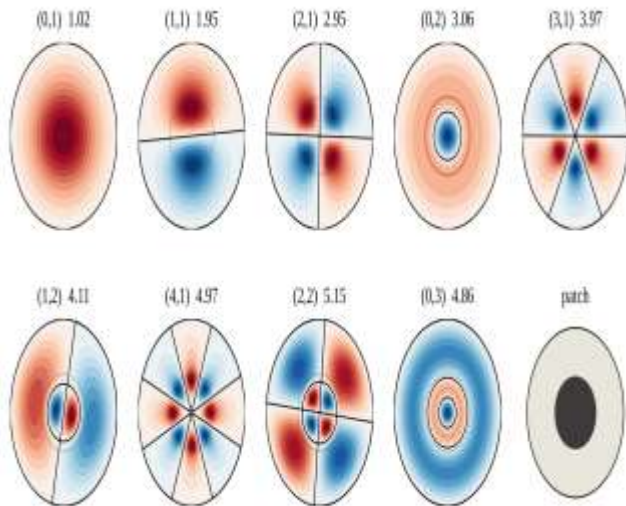


Fig. 5. Mode shapes of the nine modes identified by Raman, computed with the two-dimensional solver at the single-zone optimum. Each panel gives (m,n) and f/f_h ; solid lines are nodal lines and the dotted circle is the patch boundary.

3. Two-zone patch

On real instruments the patch is applied in many thin layers, and it is thicker at the centre than at its edge. The simplest model of such grading is a two-zone patch with density ratio α_1 for $r < b_1$ and $\alpha_2 \leq \alpha_1$ for $b_1 \leq r < b_2$:

$$\sigma(r) = \begin{cases} \alpha_1 \sigma_0, & r < b_1 \\ \alpha_2 \sigma_0, & b_1 \leq r < b_2 \\ \sigma_0, & b_2 \leq r \leq a \end{cases}$$

Optimising the four parameters with the central density ratio limited to a maximum value $\alpha_{\max}$ gives the results in Table 1. The optimum always lies on the limit, so heavier central loading always helps, but with diminishing returns: the inharmonicity is 20.4 cents for $\alpha_{\max} = 10$, 15.2 cents for $\alpha_{\max} = 20$ and 14.4 cents for $\alpha_{\max} = 40$. With $\alpha_{\max} = 20$ the inner zone extends to $0.28a$ and the outer zone to $0.46a$. Fig. 6 compares the per-mode deviations of the single-zone and two-zone optima. Grading the patch brings eight of the nine modes within 18 cents of their harmonic targets. The exception is the mode (1,1), the second harmonic, which remains about 35 cents flat in every optimised configuration, a systematic limitation of the ideal-membrane model.

Table 1. Optimised patch configurations and their inharmonicity.

Configurat ion	α_1	α_2	b_1 /a	b_2 /a	Added mass / membra ne mass	RMS (cent s)
Uniform membran e	1	–	–	–	0	216
Single zone (optimum)	8.9 1	–	0.41 4	–	1.36	37.1
Two zones, $\alpha_{\max} = 10$	10. 0	8.1 5	0.25 7	0.44 3	1.52	20.4
Two zones, $\alpha_{\max} = 20$	20. 0	14. 0	0.28 2	0.45 9	3.22	15.2
Two zones, $\alpha_{\max} = 40$	40. 0	26. 9	0.28 6	0.46 8	6.75	14.4

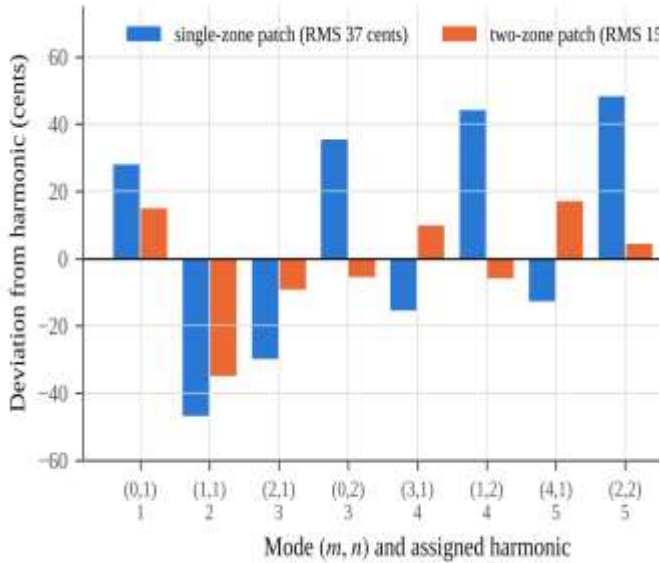


Fig. 6. Deviation of each of Raman's nine modes from its harmonic target for the optimum single-zone patch and the optimum two-zone patch ($\alpha_{\max} = 20$).

4. Sensitivity to manufacturing errors

Fig. 7 shows how the inharmonicity and the base pitch of the single-zone optimum respond to errors in patch mass (density ratio) and patch radius, varied one at a time. Harmonicity is remarkably tolerant of mass errors: errors of $\pm 5\%$ raise the RMS inharmonicity only from 37.1 to 37.6 cents. It is more sensitive to radius: errors of -5% and $+5\%$ raise it to 43.4 and 41.6 cents. The base pitch, by contrast, is sensitive to both. A 5% excess of mass lowers it by 34 cents, and a 5% excess of radius lowers it by 60 cents.

Expressed as logarithmic sensitivities, $\partial \ln f_h / \partial \ln \alpha \approx -0.40$ and $\partial \ln f_h / \partial \ln \beta \approx -0.71$. The first value lies between zero, the value for a weightless patch, and $-1/2$, the limit for a patch so heavy that it dominates the kinetic energy of every mode. The pitch can be restored by adjusting the tension, which the player does routinely, so pitch errors matter less in manufacture than errors of harmonicity.

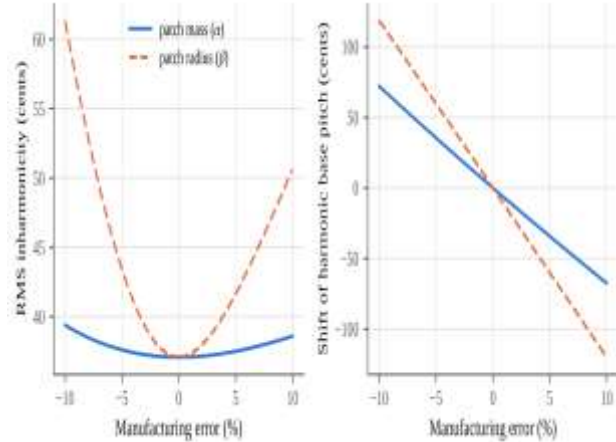


Fig. 7. Sensitivity of the single-zone optimum to manufacturing errors in patch mass (α) and patch radius (β): (left) RMS inharmonicity; (right) shift of the harmonic base pitch.

To represent a production run, 600 heads were simulated with independent, normally distributed relative errors in mass and radius:

$$\alpha = \alpha^*(1 + \varepsilon_\alpha), \quad \beta = \beta^*(1 + \varepsilon_\beta), \quad \varepsilon_\alpha, \varepsilon_\beta \sim \mathcal{N}(0, (t/2)^2)$$

where t is the tolerance interpreted as two standard deviations [20]. The left panel of Fig. 8 shows the distributions. For $t = \pm 2\%$ the median inharmonicity is 37.2 cents and 95% of heads lie below 38.0 cents, which is indistinguishable from the ideal. For $t = \pm 5\%$ the median is 37.8 cents and the 95th percentile 42.5 cents. For $t = \pm 10\%$ the median rises to 39.9 cents and the 95th percentile to 58.7 cents, so a noticeable fraction of heads would be clearly worse than the design.

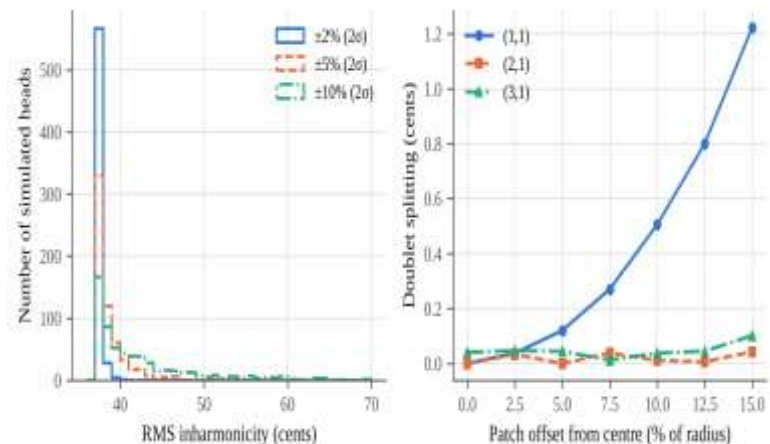


Fig. 8. (Left) Distribution of RMS inharmonicity for 600 simulated heads with random errors in patch mass and radius at three tolerance levels. (Right) Splitting of the degenerate mode pairs when the patch is displaced

from the centre, computed with the two-dimensional solver.

5. Off-centre patches

A patch that is not exactly centred breaks the circular symmetry, and each pair of modes with $m \geq 1$, which is degenerate for a centred patch, splits into two closely spaced frequencies. The right panel of Fig. 8 shows the splitting computed with the two-dimensional solver. For the (1,1) pair the splitting grows roughly as the square of the offset, reaching 0.12 cents for an offset of 5% of the radius, 0.51 cents for 10% and 1.2 cents for 15%. For the (2,1) and (3,1) pairs it remains below 0.1 cents, within the numerical resolution of the mesh. The fundamental (0,1) rises by about 7 cents relative to the harmonic base at an offset of 10% and by about 16 cents at 15%. Centring errors of a few per cent of the radius are therefore acoustically harmless. The quadratic growth has a simple explanation. Displacing the patch by a distance e in one direction or by the same distance in the opposite direction produces two heads that are mirror images of each other and therefore have identical spectra. The splitting must consequently be an even function of e , and its leading term is proportional to e^2 . For this study the two-dimensional meshes used elements of size $0.012a$ for the mode shapes and $0.008a$ for the off-centre calculations; the residual splitting of about 0.05 cents found for a centred patch measures the effect of the slight asymmetry of the mesh.

6. Comparison with earlier work

The results are consistent with the picture that emerges from earlier studies. Raman's grouping of the nine modes is reproduced: at the optimum each computed mode lies closest to the harmonic that Raman assigned to it, and the groups form progressively as the patch becomes heavier (Fig. 4). Ramakrishna and Sondhi showed with a composite-membrane analysis that a suitably loaded central region brings the overtones close to harmonic [8], and the optimum found here gives a quantitative version of their conclusion for a uniform patch. Sathej and Adhikari modelled the head with a continuously varying density and found good agreement with measured spectra [11]; the improvement obtained here with a two-zone patch points in the same direction, since a graded patch approximates their continuous profile. Gaudet and co-workers approached the problem as one of optimal

design and emphasised that the traditional instruments represent a refined solution [10]. The present contribution is to add the manufacturing dimension: the same computations that locate the optimum also show how far one may depart from it.

The inharmonicity of about 37 cents for the single-zone optimum may seem large, but it should be interpreted with care. The deviations are distributed across the nine modes, several of which share a harmonic, and the higher modes decay rapidly after the stroke. The ear judges pitch mainly from the strongest low harmonics, and the first three groups are within 47 cents of their targets even for the single-zone patch, and within 35 cents for the two-zone patch [13], [15], [21].

V. IMPLICATIONS FOR STANDARDISED MANUFACTURE

The results can be summarised as a provisional specification for a single-zone right-hand head, expressed in terms of the head radius a and the mass of the bare membrane. The patch radius should be $0.41a$ and the patch should add about 1.4 times the membrane mass, equivalent to an areal density ratio of about 9. The patch radius should be held within $\pm 2\%$ and the patch mass within $\pm 5\%$, which keeps the inharmonicity of almost all heads within about 5 cents of the design value; the larger mass tolerance is acceptable because the resulting pitch errors can be corrected by tension. Centring within 5% of the radius is sufficient. A graded patch improves harmonicity substantially, and a design in which the central third of the patch is loaded to about 20 times and the outer part to about 14 times the membrane density reduces the inharmonicity to about 15 cents.

These tolerances are within reach of simple workshop methods. Patch radius can be set with a template or a turned ring. Patch mass can be controlled by weighing the paste for each layer, or by weighing the head before and after application. Quality can be checked non-destructively by tapping the finished head, recording the sound and computing the ratios of its spectral peaks, which can then be compared with the targets of Table 1. A recording of one second gives a frequency

resolution of 1 Hz, which at 200 Hz corresponds to about 9 cents; interpolation between spectral bins improves this further, so the method can resolve the differences of a few cents that the tolerance analysis requires. Statistical process control charts applied to these measurements would allow a workshop to detect drift in its process before defective heads accumulate [20].

The finite-element approach is also a design tool. Makers already vary the composition and layering of the paste empirically. The model allows such variations to be evaluated before they are tried, and it can be extended to non-concentric or non-circular patches and to the double-layered outer ring that real heads possess, without any change to the method.

VI. LIMITATIONS

The model neglects three effects that matter on real instruments. First, the surrounding air loads the membrane and lowers the frequencies of the axisymmetric modes, especially (0,1), more than the others; studies of timpani and of Indian drums show that air loading shifts the mode frequencies appreciably and can either improve or degrade harmonicity [11], [22]. The persistent flatness of the (1,1) mode found here is the kind of discrepancy that air loading and patch stiffness could correct. Second, the patch has bending stiffness, which raises the frequencies of modes with strong curvature within the patch. Third, the right-hand head of the mridangam has an outer ring of additional membrane and is not a single uniform sheet. The optimised parameters reported here should therefore be regarded as a reference design for the idealised problem rather than as values to be copied directly onto instruments. The method, however, extends directly to all three effects, and the conclusions about relative tolerances are likely to be robust because they depend on the same mode-shape mechanisms.

VII. CONCLUSION

Raman's observation that a loaded drumhead can produce harmonic overtones has been reproduced with two independent finite-element solvers validated

against exact solutions. For an ideal membrane, a concentric patch of radius $0.414a$ and density ratio 8.9 reduces the RMS inharmonicity of Raman's nine modes from 216 cents to 37 cents, and a two-zone patch reduces it to about 15 cents. Harmonicity tolerates mass errors of $\pm 5\%$ and centring errors of several per cent, but it is more sensitive to patch radius, which should be held to about $\pm 2\%$. These results provide a quantitative basis for writing manufacturing specifications for the heads of Indian drums, and they show how a classic result of Indian physics can inform the standardisation of a traditional craft.

Declarations

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- **Conflict of interest:** The author declares no conflict of interest.
- **Data availability:** All data needed to reproduce the results are contained in the manuscript; the parameter values and equations used for every figure are stated in the text.
- **Ethics approval:** Not applicable.
- **Author contribution:** The sole author conceived the study, carried out the analysis and wrote the manuscript.

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